Rigorous Derivation of The Dirac Equation via Spectral Decomposition

By Dr. rer. nat. habil. Norbert Schwarzer

Abstract

When following the classical derivation of the Dirac equation [A1], one is confronted with the problem of the “sudden” appearance of a functional vector, being called the spinor. It is been introduced out of the need to correct a certain mismatch in rank during the decomposition of the Klein-Gordon equation. Following our recent purely metrical derivations of the Dirac equation [A2, A3, A4] we now intend to rigorously show the origin of this spinor object.

[A1] P. A. M. Dirac, “The Quantum Theory of the Electron”, Published 1 February 1928.DOI: 10.1098/rspa.1928.0023

[A2] N. Schwarzer, “Derivation of a Metric-Dimensional Based Dirac Equation”, 2026, a SIO publication, www.siomec.com

[A3] N. Schwarzer, “The Dimensional Dirac Equation with Potential”, 2026, a SIO publication, www.siomec.com

[A4] N. Schwarzer, “The Dimensional Dirac Equation with Potential and Mass”, 2026, a SIO publication, www.siomec.com

The Dimensional Dirac

In [1] our starting point was the Klein-Gordon equation:

$$0 = f_{,\alpha\beta} g^{\alpha\beta}, \tag{1}$$

which we expanded as follows:

$$0 = f_{,\alpha\beta} g^{\alpha\beta} \quad \rightarrow \quad 0 = \delta^i_j \delta^j_k f_{,\alpha\beta} g^{\alpha\beta}. \tag{2}$$

This allows us to derive:

$$0 = \delta^i_j \delta^j_k f_{,\alpha\beta} g^{\alpha\beta} = \frac{\gamma^{i\alpha}_j \partial_\alpha \gamma^{j\beta}_k \partial_\beta f + \gamma^{j\beta}_k \partial_\beta \gamma^{i\alpha}_j \partial_\alpha f}{2}. \tag{3}$$

In the last line we see that there is a summation we cannot simply tear apart:

$$0 = \frac{\gamma^{i\alpha}_{\boxed{j}} \partial_\alpha \gamma^{\boxed{j}\beta}_k \partial_\beta f + \gamma^{\boxed{j}\beta}_k \partial_\beta \gamma^{i\alpha}_{\boxed{j}} \partial_\alpha f}{2}. \tag{4}$$

Instead of just classically doing so (e.g. [2]), and then introducing or assuming something like:

$$\gamma^{i\alpha}_{\boxed{j}}\partial_\alpha\gamma^{\boxed{j}\beta}_{k}\partial_\beta f = \Gamma^{i\alpha}\partial_\alpha\gamma^{\boxed{j}\beta}_{k}\partial_\beta f_{\boxed{j}};\quad \gamma^{\boxed{j}\beta}_{k}\partial_\beta\gamma^{i\alpha}_{\boxed{j}}\partial_\alpha f = \Gamma^{\beta}_{k}\partial_\beta\gamma^{i\alpha}_{\boxed{j}}\partial_\alpha f^{\boxed{j}}$$
$$\Rightarrow \qquad , \tag{5}$$
$$0 = \frac{\gamma^{i\alpha}_{\boxed{j}}\partial_\alpha\gamma^{\boxed{j}\beta}_{k}\partial_\beta f + \gamma^{\boxed{j}\beta}_{k}\partial_\beta\gamma^{i\alpha}_{\boxed{j}}\partial_\alpha f}{2} = \frac{\Gamma^{i\alpha}\partial_\alpha\gamma^{\boxed{j}\beta}_{k}\partial_\beta f_{\boxed{j}} + \Gamma^{\beta}_{k}\partial_\beta\gamma^{i\alpha}_{\boxed{j}}\partial_\alpha f^{\boxed{j}}}{2}$$

We aim for a slightly different reasoning, namely:

$$\begin{aligned} 0 &= \gamma^{i\alpha}_{\boxed{j}}\partial_\alpha\gamma^{\boxed{j}\beta}_{k}\partial_\beta f + \gamma^{\boxed{j}\beta}_{k}\partial_\beta\gamma^{i\alpha}_{\boxed{j}}\partial_\alpha f = \not{\partial}^{i\alpha}_{\boxed{j}}\partial_\alpha\gamma^{\boxed{j}\beta}_{k}\partial_\beta f + \not{\partial}^{\boxed{j}\beta}_{k}\partial_\beta\gamma^{i\alpha}_{\boxed{j}}\partial_\alpha f \\ &= \Gamma^{i\alpha}\partial_\alpha\gamma^{\boxed{j}\beta}_{k}\partial_\beta f_{\boxed{;j}} + \Gamma^{\beta}_{k}\partial_\beta\gamma^{i\alpha}_{\boxed{j}}\partial_\alpha f^{\boxed{;j}} \end{aligned} \quad , \tag{6}$$

where we understand the j-contraction of the gamma matrices in combination with the partial derivatives $\partial_\alpha, \partial_\beta$ with respect to the ordinary coordinates as some kind of special derivative acting on f. Of course, in this sense we would result in a gradient $f_{\boxed{;j}}, f^{\boxed{;j}}$ and as with every gradient, it is a vector.

Now we can perform the split-up and obtain the usual mass-free Dirac equation:

$$\begin{aligned} 0 &= \gamma^{\boxed{j}\beta}_{k}\partial_\beta f_{\boxed{;j}} \\ 0 &= \gamma^{i\alpha}_{\boxed{j}}\partial_\alpha f^{\boxed{;j}} \end{aligned} . \tag{7}$$

How to add mass via dimensional mass creation has been shown many times in our previous publications (e.g. [4 - 7]). The same technique also holds for the incorporation of a potential (see references given in the appendix and [5 – 15]).

Alternative Way to Introduce the Dirac Matrices

One might like to see the whole Dirac theory just as one special kind of spectral decomposition of a Klein-Gordon eigenvalue problem. The unfortunate issue with the classical procedure to derive the Dirac equation, however, would then be seen in the fact that the Dirac theory does not introduce the eigenvalues as diagonal matrices, but as a vector (named a spinor). Even as only postulated, the Dirac approach simply was too successful to dismiss this little but decisive difference between Dirac and the spectral decomposition as unimportant.

Thinking about this problem, we come across the mathematical question of a rigorous incorporation of:

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{8}$$

While it is straight forward to demand the list of eigenvalues λ_i, it is absolutely not trivial to make this list a diagonal matrix. For the reason of simplicity, we want to consider a list of eigenvalues of the kind:

$$\boldsymbol{\lambda}^2 = \left\{\lambda_1^2, \lambda_2^2, \lambda_3^2, \ldots\right\}, \tag{9}$$

within a spectral decomposition problem of the type:

$$\hat{\mathrm{N}} = \mathrm{N}\boldsymbol{\lambda}^2\mathrm{N}^{\mathrm{T}} = 0\,. \tag{10}$$

The question is now, how can we make the list of eigenvalues (all being squares) to a diagonal matrix of the kind:

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} \lambda_{11} & 0 & \cdots \\ 0 & \lambda_{12} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}; \quad \boldsymbol{\lambda}_2 = \begin{pmatrix} \lambda_{21} & 0 & \cdots \\ 0 & \lambda_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{11}$$

in a rigorous manner? Obviously, the best way forward is to apply the Kronecker product with a set of suitable objects O_i assuring the right outcome as follows:

$$\boldsymbol{\lambda}\otimes\boldsymbol{\lambda} = \{\mathrm{O}_1\lambda_1, \mathrm{O}_2\lambda_2, \ldots\}\otimes\{\mathrm{O}_1\lambda_1, \mathrm{O}_2\lambda_2, \ldots\} = \begin{pmatrix} \mathrm{O}_1^2\lambda_1^2 & \mathrm{O}_1\lambda_1\mathrm{O}_2\lambda_2 & \cdots \\ \mathrm{O}_2\lambda_2\mathrm{O}_1\lambda_1 & \mathrm{O}_2^2\lambda_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{12}$$

Thereby we see that we only need the gamma matrices, adjust the product in the following way:

$$\frac{\boldsymbol{\lambda}\otimes\boldsymbol{\lambda} + (\boldsymbol{\lambda}\otimes\boldsymbol{\lambda})^{\mathrm{T}}}{2} = \frac{1}{2}\begin{pmatrix} 2\mathrm{O}_1^2\lambda_1^2 & \mathrm{O}_1\lambda_1\mathrm{O}_2\lambda_2 + \mathrm{O}_2\lambda_2\mathrm{O}_1\lambda_1 & \cdots \\ \mathrm{O}_2\lambda_2\mathrm{O}_1\lambda_1 + \mathrm{O}_1\lambda_1\mathrm{O}_2\lambda_2 & 2\mathrm{O}_2^2\lambda_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{13}$$

and obtain what we need due to:

$$\begin{aligned} \frac{\boldsymbol{\lambda}\otimes\boldsymbol{\lambda} + (\boldsymbol{\lambda}\otimes\boldsymbol{\lambda})^{\mathrm{T}}}{2} &= \frac{1}{2}\begin{pmatrix} 2\gamma_1^2\lambda_1^2 & \gamma_1\lambda_1\gamma_2\lambda_2 + \gamma_2\lambda_2\gamma_1\lambda_1 & \cdots \\ \gamma_2\lambda_2\gamma_1\lambda_1 + \gamma_1\lambda_1\gamma_2\lambda_2 & 2\gamma_2^2\lambda_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \\ &= \begin{pmatrix} \mathrm{I}\cdot\lambda_1^2 & \mathrm{I}\cdot 0 & \cdots \\ \mathrm{I}\cdot 0 & \mathrm{I}\cdot\lambda_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \mathrm{I}\cdot\begin{pmatrix} \lambda_1^2 & 0 & \cdots \\ 0 & \lambda_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \end{aligned} \tag{14}$$

Alternatively, we can also rigorously construct the (11) form from a vector of eigenvalues:

$$\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots\} \tag{15}$$

via:

$$\boldsymbol{\lambda}^2 = \boldsymbol{\lambda}\otimes\mathbf{e}\odot\mathrm{I}\cdot\boldsymbol{\lambda}\otimes\mathbf{e}\odot\mathrm{I} = \begin{pmatrix} \lambda_1^2 & 0 & \cdots \\ 0 & \lambda_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{16}$$

$$\textbf{with:} \quad \mathbf{e} = \{1,1,1,\ldots\}$$

where the operators $\otimes, \odot, \cdot$ define the Kronecker, the Hadamard, and the ordinary matrix product, respectively.

Again, the outcome would be a Dirac-like and linear equation.

Apparently, Dirac Is Not What Everyone Thought It Was

For convenience and as the following is very important, we here repeat the derivations earlier publication (e.g. [5, 6]), this time using the extended Hilbert-Action integral (e.g. [7]), reading:

$$\begin{aligned} 0 &= \int_V d^n x\sqrt{-g}\left(R_{\kappa\lambda}\delta g^{\kappa\lambda} - R\cdot\left(\frac{1}{2}+H\right)g_{\kappa\lambda}\delta g^{\kappa\lambda}\right) \\ &= \int_V d^n x\sqrt{-g}\left(R_{\kappa\lambda} - R\cdot\left(\frac{1}{2}+H\right)g_{\kappa\lambda}\right)\delta g^{\kappa\lambda} \quad . \\ &\Rightarrow R_{\kappa\lambda} - R\cdot\left(\frac{1}{2}+H\right)g_{\kappa\lambda} = 0 \end{aligned} \tag{17}$$

We saw in [7] that the complete equation, when assuming a scaled metric tensor of the form:

$$G_{\alpha\beta} = g_{\alpha\beta}\cdot F[f], \tag{18}$$

with a suitable function F[f]:

$$F[f] = C_{f1}\left(f + C_{f0}\right)^{-2}, \tag{19}$$

would read:

$$0 = \left(\frac{\left(f+C_{f0}\right)}{(n-2)}\cdot\left(\begin{array}{c} \frac{\left(f+C_{f0}\right)^2}{(n-2)}\cdot\left(g_{\alpha\beta}R\cdot\left(\frac{1}{2}+H\right) - R_{\alpha\beta}\right) \\ -\left(\begin{array}{c} f_{,\alpha\beta}(n-2) + f_{,a}g^{ab}\left(g_{\beta b,\alpha} - g_{\beta\alpha,b}\right) - f_{,\alpha}g^{ab}g_{\beta b,a} \\ -f_{,\beta}g^{ab}g_{\alpha b,a} + f_{,d}g^{cd}\left(g_{\alpha c,\beta} - \frac{1}{2}ng_{\alpha c,\beta} - \frac{1}{2}ng_{\beta c,\alpha} + \frac{1}{2}ng_{\alpha\beta,c}\right) \end{array}\right) \\ -g_{\alpha\beta}\left(f_{,ab}g^{ab} + f_{,d}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right) \\ +g_{\alpha\beta}\left(\begin{array}{c}(n-1)\left(2g^{ab}f_{,ab} + f_{,d}g^{cd}g^{ab}g_{ab,c}\right) \\ -nf_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right)\cdot\left(\frac{1}{2}+H\right) \\ -g_{\alpha\beta}f_{,c}f_{,d}g^{cd}\left(\frac{(n-1)(n+2nH-2)}{2(n-2)}\right) \end{array}\right)\right). \tag{20}$$

When investigating the situation for bigger n and realizing that for systems with high dimensionality the terms:

$$0=\left(\begin{array}{c}\dfrac{\left(\mathrm{f}+C_{\mathrm{f}0}\right)^{2}}{(\mathrm{n}-2)}\cdot\left(\mathrm{g}_{\alpha\beta}\mathrm{R}\cdot\left(\dfrac{1}{2}+H\right)-\mathrm{R}_{\alpha\beta}\right)\\ \dfrac{\left(\mathrm{f}+C_{\mathrm{f}0}\right)}{(\mathrm{n}-2)}\cdot\left(\begin{array}{c}-\left(\begin{array}{c}\mathrm{f}_{,\alpha\beta}(\mathrm{n}-2)+\mathrm{f}_{,\mathrm{a}}\mathrm{g}^{\mathrm{ab}}\left(\mathrm{g}_{\beta\mathrm{b},\alpha}-\mathrm{g}_{\beta\alpha,\mathrm{b}}\right)-\mathrm{f}_{,\alpha}\mathrm{g}^{\mathrm{ab}}\mathrm{g}_{\beta\mathrm{b},\mathrm{a}}\\ -\mathrm{f}_{,\beta}\mathrm{g}^{\mathrm{ab}}\mathrm{g}_{\alpha\mathrm{b},\mathrm{a}}+\mathrm{f}_{,\mathrm{d}}\mathrm{g}^{\mathrm{cd}}\left(\mathrm{g}_{\alpha\mathrm{c},\beta}-\dfrac{1}{2}\mathrm{n}\mathrm{g}_{\alpha\mathrm{c},\beta}-\dfrac{1}{2}\mathrm{n}\mathrm{g}_{\beta\mathrm{c},\alpha}+\dfrac{1}{2}\mathrm{n}\mathrm{g}_{\alpha\beta,\mathrm{c}}\right)\end{array}\right)\\ -\mathrm{g}_{\alpha\beta}\left(\mathrm{f}_{,\mathrm{ab}}\mathrm{g}^{\mathrm{ab}}+\mathrm{f}_{,\mathrm{d}}\mathrm{g}^{\mathrm{cd}}\dfrac{1}{2}\mathrm{g}_{\mathrm{ab},\mathrm{c}}\mathrm{g}^{\mathrm{ab}}\right)\\ +\mathrm{g}_{\alpha\beta}\left(\begin{array}{c}(\mathrm{n}-1)\left(2\mathrm{g}^{\mathrm{ab}}\mathrm{f}_{,\mathrm{ab}}+\mathrm{f}_{,\mathrm{d}}\mathrm{g}^{\mathrm{cd}}\mathrm{g}^{\mathrm{ab}}\mathrm{g}_{\mathrm{ab},\mathrm{c}}\right)\\ -\mathrm{n}\mathrm{f}_{,\mathrm{d}}\mathrm{g}^{\mathrm{cd}}\mathrm{g}^{\mathrm{ab}}\mathrm{g}_{\mathrm{ac},\mathrm{b}}\end{array}\right)\cdot\left(\dfrac{1}{2}+H\right)\end{array}\right)\end{array}\right)\tag{21}$$

become recessive, while the nonlinear term:

$$-\mathrm{g}_{\alpha\beta}\mathrm{f}_{,\mathrm{a}}\mathrm{f}_{,\mathrm{b}}\mathrm{g}^{\mathrm{ab}}\left(\frac{(\mathrm{n}-1)(\mathrm{n}+2\mathrm{n}H-2)}{2(\mathrm{n}-2)}\right)\tag{22}$$

dominates the scene.

In other words, our quantum gravity field equations:

$$0=\left(\begin{array}{c}\left(\mathrm{f}+C_{\mathrm{f}0}\right)^{2}\cdot\left(\mathrm{g}_{\alpha\beta}\mathrm{R}\cdot\left(\dfrac{1}{2}+H\right)-\mathrm{R}_{\alpha\beta}\right)\\ \left(\mathrm{f}+C_{\mathrm{f}0}\right)\cdot\left(\begin{array}{c}-\left(\begin{array}{c}\mathrm{f}_{,\alpha\beta}(\mathrm{n}-2)+\mathrm{f}_{,\mathrm{a}}\mathrm{g}^{\mathrm{ab}}\left(\mathrm{g}_{\beta\mathrm{b},\alpha}-\mathrm{g}_{\beta\alpha,\mathrm{b}}\right)-\mathrm{f}_{,\alpha}\mathrm{g}^{\mathrm{ab}}\mathrm{g}_{\beta\mathrm{b},\mathrm{a}}\\ -\mathrm{f}_{,\beta}\mathrm{g}^{\mathrm{ab}}\mathrm{g}_{\alpha\mathrm{b},\mathrm{a}}+\mathrm{f}_{,\mathrm{d}}\mathrm{g}^{\mathrm{cd}}\left(\mathrm{g}_{\alpha\mathrm{c},\beta}-\dfrac{1}{2}\mathrm{n}\mathrm{g}_{\alpha\mathrm{c},\beta}-\dfrac{1}{2}\mathrm{n}\mathrm{g}_{\beta\mathrm{c},\alpha}+\dfrac{1}{2}\mathrm{n}\mathrm{g}_{\alpha\beta,\mathrm{c}}\right)\end{array}\right)\\ -\mathrm{g}_{\alpha\beta}\left(\mathrm{f}_{,\mathrm{ab}}\mathrm{g}^{\mathrm{ab}}+\mathrm{f}_{,\mathrm{d}}\mathrm{g}^{\mathrm{cd}}\dfrac{1}{2}\mathrm{g}_{\mathrm{ab},\mathrm{c}}\mathrm{g}^{\mathrm{ab}}\right)\\ +\mathrm{g}_{\alpha\beta}\left(\begin{array}{c}(\mathrm{n}-1)\left(2\mathrm{g}^{\mathrm{ab}}\mathrm{f}_{,\mathrm{ab}}+\mathrm{f}_{,\mathrm{d}}\mathrm{g}^{\mathrm{cd}}\mathrm{g}^{\mathrm{ab}}\mathrm{g}_{\mathrm{ab},\mathrm{c}}\right)\\ -\mathrm{n}\mathrm{f}_{,\mathrm{d}}\mathrm{g}^{\mathrm{cd}}\mathrm{g}^{\mathrm{ab}}\mathrm{g}_{\mathrm{ac},\mathrm{b}}\end{array}\right)\cdot\left(\dfrac{1}{2}+H\right)\end{array}\right)\\ -\mathrm{g}_{\alpha\beta}\mathrm{f}_{,\mathrm{c}}\mathrm{f}_{,\mathrm{d}}\mathrm{g}^{\mathrm{cd}}\left(\dfrac{(\mathrm{n}-1)(\mathrm{n}+2\mathrm{n}H-2)}{2}\right)\end{array}\right).\tag{23}$$

converge to the following extremely simple, but still nonlinear result in a limiting procedure for high numbers of n:

$$0=\lim_{n\to"\infty"}\left(\frac{(f+C_{f0})}{(n-2)}\cdot\left(\begin{array}{c}\frac{(f+C_{f0})^2}{(n-2)}\cdot\left(g_{\alpha\beta}R\cdot\left(\frac{1}{2}+H\right)-R_{\alpha\beta}\right)\\ -\left(\begin{array}{c}f_{,\alpha\beta}(n-2)+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-f_{,\alpha}g^{ab}g_{\beta b,a}\\ -f_{,\beta}g^{ab}g_{\alpha b,a}+f_{,d}g^{cd}\left(\begin{array}{c}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}\\ -\frac{1}{2}ng_{\beta c,\alpha}+\frac{1}{2}ng_{\alpha\beta,c}\end{array}\right)\end{array}\right)\\ -g_{\alpha\beta}\left(f_{,ab}g^{ab}+f_{,d}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right)\\ +g_{\alpha\beta}\left(\begin{array}{c}(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)\\ -nf_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right)\cdot\left(\frac{1}{2}+H\right)\\ -2g_{\alpha\beta}f_{,a}f_{,b}g^{ab}\left(\frac{(n-1)(n+2nH-2)}{4(n-2)}\right)\end{array}\right)\right)$$

$$\Rightarrow\boxed{0=2g_{\alpha\beta}f_{,a}f_{,b}g^{ab}}\quad.\qquad(24)$$

It should be pointed out that this is the same evaluation as before, but this time we did not ignore the metric tensor $g_{\alpha\beta}$ in the term $-g_{\alpha\beta}f_{,a}f_{,b}g^{ab}\left(\frac{(n-1)(n+2nH-2)}{2(n-2)}\right)$. Now we remember what was said previously (see e.g. [7], sections "Towards Fermionic "Coherent Domains", Part I: Linearity via Spectral-Operator Decomposition" and "Towards Fermionic "Coherent Domains", Part II:…"), where we observed the product $g_{\alpha\beta}g^{ab}$ and realized that this is very close to the metric definition of the classical Dirac matrices usually given as:

$$g^{\alpha\beta}\cdot I=\frac{\gamma^{\alpha}\gamma^{\beta}+\gamma^{\beta}\gamma^{\alpha}}{2};\quad g_{\alpha\beta}\cdot I=\frac{\gamma_{\alpha}\gamma_{\beta}+\gamma_{\beta}\gamma_{\alpha}}{2},\qquad(25)$$

only that here, we do not need to artificially weave in the unit matrix I, but either have to adjust the Dirac matrices a bit, so that they could reproduce something like:

$$g_{\mu\nu}g^{ab}=\left[\frac{\gamma^{a}\gamma^{b}+\gamma^{b}\gamma^{a}}{2}\right]_{\mu\nu}=\frac{\gamma^{a}\gamma^{b}+\gamma^{b}\gamma^{a}}{2},\qquad(26)$$

or extract the necessary unit matrix from the covariant metric tensor as follows:

$$g_{\mu\nu}g^{ab}f_{,ab}=g_{\mu\lambda}\delta^{\lambda}_{\nu}g^{ab}f_{,ab}=g_{\mu\lambda}\cdot I\cdot g^{ab}f_{,ab}.\qquad(27)$$

With the first option (if at all possible; see [16]), the last line in (24) could be transformed to:

$$\begin{aligned}0&=2g_{\alpha\beta}f_{,a}f_{,b}g^{ab}=g_{\alpha\beta}g^{ab}f_{,a}f_{,b}+g_{\alpha\beta}g^{ab}f_{,a}f_{,b}\\&=\left(\gamma^{a}\gamma^{b}+\gamma^{b}\gamma^{a}\right)f_{,a}f_{,b}=\gamma^{a}f_{,a}f_{,b}\gamma^{b}+\gamma^{b}f_{,a}f_{,b}\gamma^{a}\end{aligned}.\qquad(28)$$

However, as we have already seen in [7], section "Towards Fermionic "Coherent Domains", Part II:…", the much simpler path forward is to apply (27). In doing so, we obtain from (24) (last line):

$$2g_{\alpha\beta}f_{,a}f_{,b}g^{ab}=2g_{\alpha\lambda}\delta^{\lambda}_{\beta}f_{,a}f_{,b}g^{ab}=2g_{\alpha\lambda}\cdot I\cdot f_{,a}f_{,b}g^{ab},\qquad(29)$$

where with $I \cdot g^{ab}$ we exactly have what we need in order to apply (25). Then the rank has not been artificially changed as it was the case within the Dirac theory and we can move on as follows:

$$g_{\alpha\lambda}\hat{N}^{\lambda}_{\beta}\left[F, g_{ab}\right] \equiv g_{\alpha\lambda} \cdot I \cdot f_{,a} f_{,b} g^{ab}. \tag{30}$$

The spinor would then be obtained via the usual spectral decomposition as follows:

$$g_{\alpha\lambda}\hat{N}^{\lambda}_{\beta}\left[F, g_{ab}\right] \equiv g_{\alpha\lambda} \cdot I \cdot f_{,a} f_{,b} g^{ab} = g_{\alpha\lambda} N \boldsymbol{\lambda} N^{T}, \tag{31}$$

whereby – as before – we have the eigenvectors $\mathbf{u}_i$ of the operator $\hat{N}^{\lambda}_{\beta}$ forming our N and N^T as follows:

$$N = \begin{pmatrix} \leftarrow & \mathbf{u}_1 & \rightarrow \\ \leftarrow & \mathbf{u}_2 & \rightarrow \\ \leftarrow & \vdots & \rightarrow \end{pmatrix}; \quad N^{T} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots \\ \downarrow & \downarrow & \downarrow \end{pmatrix}, \tag{32}$$

and the eigenvalues λ_i constructing the matrix $\boldsymbol{\lambda}$ via:

$$\boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 & 0 & \cdots \\ 0 & \lambda_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{33}$$

A Simple Generalization of the Spectral Decomposition and the Introduction of the Dirac Matrices

Apparently, nothing hinders us to repeat the evaluation with the Dirac matrices and write:

$$g_{\alpha\lambda}\sum_{\forall a,b}\hat{N}^{\lambda}_{\beta}\left[F, g_{ab}\right] \equiv g_{\alpha\lambda} \cdot I \cdot f_{,a} f_{,b} g^{ab} = g_{\alpha\lambda}\sum_{\forall a,b}\left(N_a \boldsymbol{\lambda} \left(N_b\right)^{T} + N_b \boldsymbol{\lambda} \left(N_a\right)^{T}\right), \tag{34}$$

where we at first had to extend the recipe and introduced a sum of matrix operation instead of just one matrix. From there we can now write the following equation:

$$\begin{aligned} \sum_{\forall a,b}\hat{N}^{\lambda}_{\beta}\left[F, g_{ab}\right] &\equiv \frac{\gamma^a\gamma^b + \gamma^b\gamma^a}{2} f_{,a} f_{,b} \\ = \sum_{\forall a,b}\left(N_a \boldsymbol{\lambda} \left(N_b\right)^{T} + N_b \boldsymbol{\lambda} \left(N_a\right)^{T}\right) &= \frac{\gamma^a\gamma^b + \gamma^b\gamma^a}{2} \partial_a f \cdot \partial_b f \\ = \frac{\gamma^a \partial_a f \cdot \partial_b f \gamma^b + \gamma^b \partial_b f \cdot \partial_a f \gamma^a}{2} &= 0 \\ \Rightarrow \quad \gamma^a \partial_a f \cdot \partial_b f \gamma^b + \gamma^b \partial_b f \cdot \partial_a f \gamma^a &= 0 \end{aligned}, \tag{35}$$

which we can spectrally decompose as follows:

$$\overbrace{\gamma^a \partial_a f \cdot \partial_b f \gamma^b}^{N_1 \overset{\lambda}{\beta}} + \overbrace{\gamma^b \partial_b f \cdot \partial_a f \gamma^a}^{N_2 \overset{\lambda}{\beta}} = \sum_{\forall a,b}\left(N_{1a} \boldsymbol{\lambda}_1 N_{1b}^{T} + N_{2b} \boldsymbol{\lambda}_2 N_{2a}^{T}\right) = 0. \tag{36}$$

This brings in the spinors as eigenvalues:

$$\boldsymbol{\lambda}_1 = \begin{pmatrix} \lambda_{11} & 0 & \cdots \\ 0 & \lambda_{12} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}; \quad \boldsymbol{\lambda}_2 = \begin{pmatrix} \lambda_{21} & 0 & \cdots \\ 0 & \lambda_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{37}$$

and shows us the Dirac theory as a special case of the spectral decomposition technique.

The only problem we have could be extracted from the fact that the Dirac matrices are not perfectly following the transposition condition, which is to say, we do not have

$$\mathrm{N} = \mathrm{N}^{\mathrm{T}}, \tag{38}$$

but still find no ruling like (please see the red symbols, which we used to point out the problem):

$$\overbrace{\gamma^{a}\partial_{a}f\cdot\partial_{b}f\left(\gamma^{b}\right)^{T}}^{N_{1_\beta}^{\lambda}} + \overbrace{\gamma^{b}\partial_{b}f\cdot\partial_{a}f\left(\gamma^{a}\right)^{T}}^{N_{2_\beta}^{\lambda}} = \mathrm{N}_1\boldsymbol{\lambda}_1\mathrm{N}_1^{\mathrm{T}} + \mathrm{N}_2\boldsymbol{\lambda}_2\mathrm{N}_2^{\mathrm{T}} = 0\,. \tag{39}$$

Knowing, however, that the Dirac matrices fulfill the conditions:

$$\gamma^0 = \left(\gamma^0\right)^{\mathrm{T}}; \quad \gamma^{\mathrm{k}} = -\left(\gamma^{\mathrm{k}}\right)^{\mathrm{T}}; \quad \mathrm{k} = 1,2,3\,, \tag{40}$$

where we used the matrix representation:

$$\gamma_{\mathrm{D}}{}^{0} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}; \quad \gamma_{\mathrm{D}}{}^{1} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix},$$
$$\gamma_{\mathrm{D}}{}^{2} = \begin{pmatrix} & & & -\mathrm{i} \\ & & \mathrm{i} & \\ & \mathrm{i} & & \\ -\mathrm{i} & & & \end{pmatrix}; \quad \gamma_{\mathrm{D}}{}^{3} = \begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}; \quad \mathrm{I} = \delta_{\mathrm{ij}} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{41}$$

we would only end up with the following combinations in (39):

$$\begin{aligned}
&\mathrm{a} = \mathrm{b} = 0: && \gamma^{a}\partial_{a}f\cdot\partial_{b}f\left(\gamma^{b}\right)^{T} + \gamma^{b}\partial_{b}f\cdot\partial_{a}f\left(\gamma^{a}\right)^{T} = \gamma^{a}\partial_{a}f\cdot\partial_{b}f\gamma^{b} + \gamma^{b}\partial_{b}f\cdot\partial_{a}f\gamma^{a} \\
&\mathrm{a} = \mathrm{b} = \mathrm{k}: && \gamma^{a}\partial_{a}f\cdot\partial_{b}f\left(\gamma^{b}\right)^{T} + \gamma^{b}\partial_{b}f\cdot\partial_{a}f\left(\gamma^{a}\right)^{T} = -\gamma^{a}\partial_{a}f\cdot\partial_{b}f\gamma^{b} - \gamma^{b}\partial_{b}f\cdot\partial_{a}f\gamma^{a} \\
&\mathrm{a} = 0; \quad \mathrm{b} = \mathrm{k}: && \gamma^{a}\partial_{a}f\cdot\partial_{b}f\left(\gamma^{b}\right)^{T} + \gamma^{b}\partial_{b}f\cdot\partial_{a}f\left(\gamma^{a}\right)^{T} = -\gamma^{a}\partial_{a}f\cdot\partial_{b}f\gamma^{b} + \gamma^{b}\partial_{b}f\cdot\partial_{a}f\gamma^{a} \\
&\mathrm{a} = \mathrm{k}; \quad \mathrm{b} = 0: && \gamma^{a}\partial_{a}f\cdot\partial_{b}f\left(\gamma^{b}\right)^{T} + \gamma^{b}\partial_{b}f\cdot\partial_{a}f\left(\gamma^{a}\right)^{T} = \gamma^{a}\partial_{a}f\cdot\partial_{b}f\gamma^{b} - \gamma^{b}\partial_{b}f\cdot\partial_{a}f\gamma^{a}
\end{aligned}. \tag{42}$$

This can also easily be proven in a very general manner and for all possible Dirac matrices, by using the fundamental Dirac bases, where we find:

$$\sigma_0=\rho_0=I=\delta_{ij}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=(\sigma_0)^T=(\rho_0)^T;\quad \sigma_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}=(\sigma_1)^T$$

$$\sigma_2=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}=-(\sigma_2)^T;\quad \sigma_3=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}=(\sigma_3)^T$$

$$\rho_1=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}=(\rho_1)^T;\quad \rho_2=\begin{pmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{pmatrix}=-(\rho_2)^T$$

$$\rho_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}=(\rho_3)^T \qquad . \quad (43)$$

Consequently, we only have a problem when the indices a and b are unequal. Then the Dirac approach “violates” the decomposition procedure. This, however, is just the case where the Dirac matrices cancel each other out anyway and so, no harm is done. The Dirac procedure, de facto, is just a spectral matrix or rather an operator-matrix decomposition.

So we have to conclude, nothing physical forces the spinors to come forth, but a formal mathematical procedure.

We are going to further investigate and discuss the decomposition elsewhere [16].

The Metric Dirac in the Case of Infinite Orthogonal Dimensionality

Taking the high dimensionality outcome with the last line of (24) and assuming a dimensional mass production as shown in [7], subsection “The Nonlinear Term Prevails in Infinity”, we already have shown how to factorize equations like:

$$0=I\cdot f_{,a}f_{,b}g^{ab}. \quad (44)$$

Following the path from [7], subsection “The Nonlinear Term Prevails in Infinity” and introducing a function f as a product f=h[…]*p[xx] plus demanding p=p[xx] with:

$$p_{,xx}p_{,xx}=M^2\cdot p^2, \quad (45)$$

while h does not depend on xx and thus:

$$f_{,xx}f_{,xx}=h^2\cdot M^2\cdot p^2, \quad (46)$$

consequently (44) would look as follows:

$$0 = \mathrm{I} \cdot \left(\mathrm{h}_{,\mathrm{a}} \mathrm{h}_{,\mathrm{b}} \mathrm{g}^{\mathrm{ab}} + \mathrm{M}^2 \mathrm{h}^2 \right). \tag{47}$$

The coordinate xx is to be considered thereby as an additional degree of freedom or dimension, which here codes the masses M.

The remaining sub-metric for our- to pick an example - spherical coordinates t, r, ϑ, φ would then read:

$$\mathrm{g}^{\mathrm{ab}} = \begin{pmatrix} -\frac{1}{\mathrm{c}^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\mathrm{r}^2} & 0 \\ 0 & 0 & 0 & \frac{1}{\mathrm{r}^2 \sin^2 \vartheta} \end{pmatrix} \tag{48}$$

and it cannot be pointed out enough that this only is the sub-metric of an otherwise pretty highly dimensional space-time… only that the function h (the wave function) does not depend on the other coordinates.

Assuming n being the number of classical Dirac dimensions, then our dimension xx increases the total number of dimensions to n+1. Applying (35), this can be expanded and we obtain[1]:

$$\begin{aligned} 0 &= 2\left(\mathrm{h}_{,\mathrm{a}} \mathrm{h}_{,\mathrm{b}} \mathrm{g}^{\mathrm{ab}} \cdot \mathrm{I} + \mathrm{M}^2 \mathrm{h}^2 \cdot \mathrm{I} \right) = \left(\gamma^{\mathrm{a}} \gamma^{\mathrm{b}} + \gamma^{\mathrm{b}} \gamma^{\mathrm{a}} \right) \mathrm{h}_{,\mathrm{a}} \mathrm{h}_{,\mathrm{b}} + 2\mathrm{M}^2 \mathrm{h}^2 \cdot \mathrm{I} \\ &= \gamma^{\mathrm{a}} \mathrm{h}_{,\mathrm{a}} \mathrm{h}_{,\mathrm{b}} \gamma^{\mathrm{b}} + \mathrm{M}^2 \mathrm{h}^2 \cdot \mathrm{I} + \gamma^{\mathrm{b}} \mathrm{h}_{,\mathrm{b}} \mathrm{h}_{,\mathrm{a}} \gamma^{\mathrm{a}} + \mathrm{M}^2 \mathrm{h}^2 \cdot \mathrm{I} \\ &= \begin{bmatrix} \left(\gamma^{\mathrm{a}} \mathrm{h}_{,\mathrm{a}} + \mathrm{i} \cdot \mathrm{M} \cdot \mathrm{h} \cdot \mathrm{I} \right) \left(\mathrm{h}_{,\mathrm{b}} \gamma^{\mathrm{b}} - \mathrm{i} \cdot \mathrm{M} \cdot \mathrm{h} \cdot \mathrm{I} \right) \\ + \left(\gamma^{\mathrm{b}} \mathrm{h}_{,\mathrm{b}} + \mathrm{i} \cdot \mathrm{M} \cdot \mathrm{h} \cdot \mathrm{I} \right) \left(\mathrm{h}_{,\mathrm{a}} \gamma^{\mathrm{a}} - \mathrm{i} \cdot \mathrm{M} \cdot \mathrm{h} \cdot \mathrm{I} \right) \end{bmatrix} \end{aligned} . \tag{49}$$

Now, the remaining task seems to interpret (49) as special (operational) spectral decomposition and to incorporate a list of eigen solutions of functions h→**h** in the form:

$$\mathbf{h} = \begin{pmatrix} \mathrm{h}_1 & 0 & \cdots \\ 0 & \mathrm{h}_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{50}$$

in order to make the decomposition complete, so that in the end we have reproduced the Dirac-like equation:

$$0 = \begin{cases} \gamma^{\mathrm{a}} \mathbf{h}_{,\mathrm{a}} - \mathrm{i} \cdot \mathrm{M} \cdot \mathbf{h} \\ \mathbf{h}_{,\mathrm{b}} \gamma^{\mathrm{b}} + \mathrm{i} \cdot \mathrm{M} \cdot \mathbf{h} \end{cases}. \tag{51}$$

This is in difference to the Dirac approach where the list of f or h has only been postulated and then named spinors.

Please note that within our derivation above, we have already incorporated the simplification:

$$\mathrm{i} \cdot \mathrm{M} \cdot \mathbf{h} \cdot \mathrm{I} = \mathrm{i} \cdot \mathrm{M} \cdot \mathbf{h} \,. \tag{52}$$

---

[1] Please note that, when understanding n to be the number of all dimensions, the summation in the expression $\mathrm{h}_{,\mathrm{a}} \mathrm{h}_{,\mathrm{b}} \mathrm{g}^{\mathrm{ab}}$ below only runs from 0 to n-2.

Now using spherical coordinates and our simple example with f=f[t, r, xx]=h[t, r]*p[xx] plus Dirac matrices as follows:

$$\gamma^0 = \frac{i}{c}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}; \quad \gamma^1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$$

$$\gamma^2 = \frac{1}{r}\begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix}; \quad \gamma^3 = \frac{1}{r\cdot\sin\vartheta}\begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}; \quad I = \delta_{ij} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad (53)$$

results in the following matrix equation from (51):

$$0 = \gamma^0 \mathbf{h}_{,0} + \gamma^1 \mathbf{h}_{,1} \pm i\cdot M\cdot \mathbf{h} = \gamma^t \mathbf{h}_{,t} + \gamma^r \mathbf{h}_{,r} \pm i\cdot M\cdot \mathbf{h}$$

$$= \frac{i}{c}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\mathbf{h}_{,t} + \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}\mathbf{h}_{,r} \pm i\cdot M\cdot \mathbf{h}$$

$$= \begin{pmatrix} \frac{i}{c}\begin{pmatrix} h_{1,t} & & & \\ & h_{2,t} & & \\ & & -h_{3,t} & \\ & & & -h_{4,t} \end{pmatrix} + \begin{pmatrix} & & & h_{4,r} \\ & & h_{3,r} & \\ & -h_{2,r} & & \\ -h_{1,r} & & & \end{pmatrix} \\ \pm i\cdot M\cdot \begin{pmatrix} h_1 & & & \\ & h_2 & & \\ & & h_3 & \\ & & & h_4 \end{pmatrix} \end{pmatrix}. \quad (54)$$

We immediately see that when taking **h** in the form (51) we would end up with a meaningful equation only for the coordinate t, where we still get the matter- and the antimatter solutions. Otherwise, the system of equations is meaningless. When interpreting **h** as vector, however, we obtain the usual Dirac-like system as follows:

$$0 = \gamma^0 \mathbf{h}_{,0} + \gamma^1 \mathbf{h}_{,1} \pm i\cdot M\cdot \mathbf{h} = \gamma^t \mathbf{h}_{,t} + \gamma^r \mathbf{h}_{,r} \pm i\cdot M\cdot \mathbf{h}$$

$$= \frac{i}{c}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\mathbf{h}_{,t} + \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}\mathbf{h}_{,r} \pm i\cdot M\cdot \mathbf{h}. \quad (55)$$

$$= \frac{i}{c}\begin{pmatrix} h_{1,t} \\ h_{2,t} \\ -h_{3,t} \\ -h_{4,t} \end{pmatrix} + \begin{pmatrix} h_{4,r} \\ h_{3,r} \\ -h_{2,r} \\ -h_{1,r} \end{pmatrix} \pm i\cdot M\cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}$$

Thus, the vector-interpretation would be the better one, not only because it obviously makes much more sense, but as it also agrees with the classical Dirac apparatus; but, unfortunately, the spectral decomposition contains no vector but a diagonal matrix of the form (50).

So, we are in a kind of dilemma and the solution to this can be found in our derivation from [7], subsection “The Alternative Way to Introduce the Dirac Matrices”.

In order to allow for this solution to unfold, we go back to our equation (47), but this time incorporate the diagonal matrix of eigen solutions straight away leading to:

$$0 = 2\left(\mathbf{h}_{,a}\mathbf{h}_{,b}g^{ab} + M^2\mathbf{h}^2\right). \tag{56}$$

Thereby, when taking the procedure of diagonal matrix construction from a vector, derived as follows:

$$\boldsymbol{\lambda}^2 = \boldsymbol{\lambda}\otimes\mathbf{e}\odot\mathrm{I}\cdot\boldsymbol{\lambda}\otimes\mathbf{e}\odot\mathrm{I} = \begin{pmatrix} \lambda_1^2 & 0 & \cdots \\ 0 & \lambda_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{57}$$
$$\textbf{with:}\quad \mathbf{e} = \{1,1,1,\ldots\}$$

where the operators $\otimes, \odot, \cdot$ define the Kronecker, the Hadamard, and the ordinary matrix product, respectively, and defining:

$$\mathbf{h}^{\mathrm{i}}_{\mathrm{j}} = \begin{pmatrix} \mathrm{h}_1 & 0 & \cdots \\ 0 & \mathrm{h}_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}; \quad \mathbf{h} = \{\mathrm{h}_1, \mathrm{h}_2, \ldots\}, \tag{58}$$

the operation between the various terms can only be as follows:

$$\begin{aligned} 0 &= 2\left(\mathbf{h}^{\mathrm{i}}_{\mathrm{j},a}\mathbf{h}^{\mathrm{j}}_{\mathrm{k},b}g^{ab} + M^2\mathbf{h}^{\mathrm{i}}_{\mathrm{j}}\mathbf{h}^{\mathrm{j}}_{\mathrm{k}}\right) \\ &= \left(\gamma^a\gamma^b + \gamma^b\gamma^a\right)\mathbf{h}_{,a}\otimes\mathbf{e}\odot\mathrm{I}\cdot\mathbf{h}_{,b}\otimes\mathbf{e}\odot\mathrm{I} + 2M^2\mathbf{h}\otimes\mathbf{e}\odot\mathrm{I}\cdot\mathbf{h}\otimes\mathbf{e}\odot\mathrm{I} \\ &= \begin{bmatrix} \gamma^a\mathbf{h}_{,a}\otimes\mathbf{e}\odot\mathrm{I}\cdot\mathbf{h}_{,b}\otimes\mathbf{e}\odot\mathrm{I}\gamma^b + M^2\mathbf{h}\otimes\mathbf{e}\odot\mathrm{I}\cdot\mathbf{h}\otimes\mathbf{e}\odot\mathrm{I} \\ +\gamma^b\mathbf{h}_{,b}\otimes\mathbf{e}\odot\mathrm{I}\cdot\mathbf{h}_{,a}\otimes\mathbf{e}\odot\mathrm{I}\,\gamma^a + M^2\mathbf{h}\otimes\mathbf{e}\odot\mathrm{I}\cdot\mathbf{h}\otimes\mathbf{e}\odot\mathrm{I} \end{bmatrix} \\ &= \begin{bmatrix} \left(\gamma^a\mathbf{h}_{,a}\otimes\mathbf{e}\odot\mathrm{I} + \mathrm{i}\cdot M\cdot\mathbf{h}\otimes\mathbf{e}\odot\mathrm{I}\right)\left(\mathbf{h}_{,b}\otimes\mathbf{e}\odot\mathrm{I}\gamma^b - \mathrm{i}\cdot M\cdot\mathbf{h}\otimes\mathbf{e}\odot\mathrm{I}\right) \\ +\left(\gamma^b\mathbf{h}_{,b}\otimes\mathbf{e}\odot\mathrm{I} + \mathrm{i}\cdot M\cdot\mathbf{h}\otimes\mathbf{e}\odot\mathrm{I}\right)\left(\mathbf{h}_{,a}\otimes\mathbf{e}\odot\mathrm{I}\gamma^a - \mathrm{i}\cdot M\cdot\mathbf{h}\otimes\mathbf{e}\odot\mathrm{I}\right) \end{bmatrix} \\ &= \begin{bmatrix} \left(\gamma^a\mathbf{h}_{,a} + \mathrm{i}\cdot M\cdot\mathbf{h}\right)\otimes\mathbf{e}\odot\mathrm{I}\left(\mathbf{h}_{,b}\gamma^b - \mathrm{i}\cdot M\cdot\mathbf{h}\right)\otimes\mathbf{e}\odot\mathrm{I} \\ +\left(\gamma^b\mathbf{h}_{,b} + \mathrm{i}\cdot M\cdot\mathbf{h}\right)\otimes\mathbf{e}\odot\mathrm{I}\left(\mathbf{h}_{,a}\gamma^a - \mathrm{i}\cdot M\cdot\mathbf{h}\right)\otimes\mathbf{e}\odot\mathrm{I} \end{bmatrix} \end{aligned}. \tag{59}$$

This leads us directly to (51), with **h** being defined via the second equation in (58) and thus, to a clear Dirac-like equation.

And why only “Dirac-like”?

Well, because the origin before the operational factorization is not the Laplace operator as in the classical case [2], but the nonlinear term in (56).

The interesting aspect here lays in the fact that the spherical metric does not show in the equation before there is no h-functional dependency on the angles. With h=h[t, r, ϑ, φ], however, (55) changes to:

$$
\begin{aligned}
0 &= \gamma^0 \mathbf{h}_{,0} + \gamma^1 \mathbf{h}_{,1} + \gamma^2 \mathbf{h}_{,2} + \gamma^3 \mathbf{h}_{,3} \pm i \cdot M \cdot \mathbf{h} \\
&= \gamma^t \mathbf{h}_{,t} + \gamma^r \mathbf{h}_{,r} + \gamma^\vartheta \mathbf{h}_{,\vartheta} + \gamma^\varphi \mathbf{h}_{,\varphi} \pm i \cdot M \cdot \mathbf{h} \\
&= \begin{pmatrix} \frac{i}{c} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \mathbf{h}_{,t} + \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \mathbf{h}_{,r} \\ \frac{1}{r} \begin{pmatrix} & & -i & \\ & i & & \\ i & & & \\ -i & & & \end{pmatrix} \mathbf{h}_{,\vartheta} + \frac{1}{r \cdot \sin\vartheta} \begin{pmatrix} & 1 & & \\ & & -1 & \\ -1 & & & \\ & 1 & & \end{pmatrix} \mathbf{h}_{,\varphi} \pm i \cdot M \cdot \mathbf{h} \end{pmatrix} \\
&= \frac{i}{c} \begin{pmatrix} h_{1,t} \\ h_{2,t} \\ -h_{3,t} \\ -h_{4,t} \end{pmatrix} + \begin{pmatrix} h_{4,r} \\ h_{3,r} \\ -h_{2,r} \\ -h_{1,r} \end{pmatrix} + \frac{i}{r} \begin{pmatrix} -h_{4,\vartheta} \\ h_{3,\vartheta} \\ h_{2,\vartheta} \\ -h_{1,\vartheta} \end{pmatrix} + \frac{1}{r \cdot \sin\vartheta} \begin{pmatrix} h_{3,\varphi} \\ -h_{4,\varphi} \\ -h_{1,\varphi} \\ h_{2,\varphi} \end{pmatrix} \pm i \cdot M \cdot \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}.
\end{aligned}
\tag{60}
$$

We leave the investigation of this system of differential equations for the interested reader.

The Metric Dirac in the Classical Form

Only assuming an f-dependency of f[t,r], with t denoting the time and r denoting the spherical radius, respectively, plus introducing the mass term in a dimensional manner as shown in the subsection above (see text following equation (44)), it follows from the linearized (weak gravity) field equation [7]:[2]

$$
\hat{N}_{\mu\nu}\left[F, g_{ab}\right] \equiv \frac{F'}{2}\left((n-1)\left(2g^{ab}f_{,ab} + f_{,d}g^{cd}g^{ab}g_{ab,c}\right) - nf_{,d}g^{cd}g^{ab}g_{ac,b}\right)g_{\mu\nu} = 0\,. \tag{61}
$$

where, for simplicity we have assumed sub-metrics of the type:

$$
g_{ij} = \begin{pmatrix} g_{00} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_{n-1n-1} \end{pmatrix}; \quad g_{ii,i} = 0\,) \tag{62}
$$

[2] Please note that, when understanding n to be the number of all dimensions, the symbols i, j run from i,j=0,1,…,n-1, we have a,b=0,1,2,…,n-2.

$$\sum_{\forall i,j}\hat{N}_{v}^{\lambda}\left[F,g_{ij}\right]\equiv C_{F}\cdot\frac{4\cdot(n-1)}{n-2}\left(f+C_{f}\right)^{\frac{6-n}{n-2}}\Delta f=$$
$$C_{F}\cdot\frac{4\cdot(n-1)}{n-2}\left(f+C_{f}\right)^{\frac{6-n}{n-2}}\left(\Delta h+M^{2}h\right)p=0$$
$$\Rightarrow\quad 0=\overbrace{C_{F}\cdot\frac{2\cdot(n-1)}{n-2}\left(f+C_{f}\right)^{\frac{6-n}{n-2}}}^{Q}\left(\Delta h+M^{2}h\right)$$
$$=Q\left(\Delta+M^{2}\right)h \qquad (63)$$

Now, we assume this to be developed into a spectral decomposition expression, introduce the diagonal eigenvalue matrix for the list of functions h as before, thereby applying (16) and (58), and obtain:

$$0=Q\left(\Delta+M^{2}\right)\mathbf{h}\otimes\mathbf{e}\odot I$$
$$=Q\left(\frac{\gamma^{a}\gamma^{b}+\gamma^{b}\gamma^{a}}{2}\partial_{a}\partial_{b}+M^{2}\right)\mathbf{h}\otimes\mathbf{e}\odot I$$
$$=\frac{1}{2}Q\left(\gamma^{a}\partial_{a}\partial_{b}\gamma^{b}+\gamma^{b}\partial_{a}\partial_{b}\gamma^{a}+2M^{2}\right)\mathbf{h}\otimes\mathbf{e}\odot I \qquad (64)$$
$$=\frac{Q}{2}\left[\gamma^{a}\partial_{a}\partial_{b}\gamma^{b}+M^{2}+\gamma^{b}\partial_{a}\partial_{b}\,\gamma^{a}+M^{2}\right]\mathbf{h}\otimes\mathbf{e}\odot I$$
$$=\frac{Q}{2}\left[\left(\gamma^{a}\partial_{a}+i\cdot M\right)\left(\partial_{b}\gamma^{b}-i\cdot M\right)+\left(\gamma^{b}\partial_{b}+i\cdot M\right)\left(\partial_{a}\gamma^{a}-i\cdot M\right)\right]\mathbf{h}\otimes\mathbf{e}\odot I$$

Here, the symbol ∂ points out the fact that we have no ordinary derivative, but one which sums up to the Laplace operator or – in case the sub-metric is not of type (62) – to the operator given in (61). As this can be simplified to:

$$0=\left[\left(\gamma^{a}\partial_{a}+i\cdot M\right)\left(\partial_{b}\gamma^{b}-i\cdot M\right)+\left(\gamma^{b}\partial_{b}+i\cdot M\right)\left(\partial_{a}\gamma^{a}-i\cdot M\right)\right]\mathbf{h}\,, \qquad (65)$$

and we see that any intrinsic solution would solve the whole equation, we can directly extract the classical Dirac equations:

$$0=\begin{cases}\gamma^{a}\partial_{a}\mathbf{h}-i\cdot M\cdot\mathbf{h}\\ \gamma^{b}\partial_{b}\mathbf{h}+i\cdot M\cdot\mathbf{h}\end{cases}. \qquad (66)$$

We refer to the textbooks[3] with respect to the usual treatment of this Dirac equation in spherical coordinates as we here want to point out only a few aspects.

In our case of the spherical coordinates with f=f[t,r], (66) results in the following vector-equation:

[3] It should be pointed out that there – which is to say in those textbooks – the problem is usually treated completely algebraically with pairs of solutions of the hydrogen Schrödinger equation.

$$0=\gamma^0\mathbf{h}_{,0}+\gamma^1\partial_1\mathbf{h}+\gamma^2\partial_2\mathbf{h}\pm i\cdot M\cdot\mathbf{h}=\gamma^t\mathbf{h}_{,t}+\gamma^r\left(\frac{1}{r}\mathbf{h}+\mathbf{h}_{,r}\right)+\gamma^\vartheta\partial_\vartheta\mathbf{h}\pm i\cdot M\cdot\mathbf{h}$$

$$=\begin{pmatrix}\frac{i}{c}\begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}\mathbf{h}_{,t}+\begin{pmatrix}&&&1\\&&1&\\&-1&&\\-1&&&\end{pmatrix}\left(\frac{1}{r}\mathbf{h}+\mathbf{h}_{,r}\right)\\+\frac{i}{2r\sin\vartheta}\begin{pmatrix}&&&-1\\&&1&\\&1&&\\-1&&&\end{pmatrix}\mathbf{h}\pm i\cdot M\cdot\mathbf{h}\end{pmatrix}$$

$$=\frac{i}{c}\begin{pmatrix}h_{1,t}\\h_{2,t}\\-h_{3,t}\\-h_{4,t}\end{pmatrix}+\begin{pmatrix}h_{4,r}+\frac{h_4}{r}-\frac{i}{2r\sin\vartheta}h_4\\h_{3,r}+\frac{h_3}{r}+\frac{i}{2r\sin\vartheta}h_3\\-h_{2,r}-\frac{h_2}{r}+\frac{i}{2r\sin\vartheta}h_2\\-h_{1,r}-\frac{h_1}{r}-\frac{i}{2r\sin\vartheta}h_1\end{pmatrix}\pm i\cdot M\cdot\begin{pmatrix}h_1\\h_2\\h_3\\h_4\end{pmatrix}\quad.\qquad(67)$$

We see that it seems that solutions with a complete angle-independency are not possible. This is in contrast to our Dirac-like equation from the subsection above for high dimensionality.

Insisting on a pure time and radius dependency, one needs to go to cylindrical coordinates where we have h=h[t, r, φ, z], and the subsequent Dirac equation from our completely metric origin (63) reads:

$$
\begin{aligned}
0 &= \gamma^0 \mathbf{h}_{,0} + \gamma^1 \partial_1 \mathbf{h} + \gamma^2 \partial_2 \mathbf{h} + \gamma^3 \partial_3 \mathbf{h} \pm \mathrm{i} \cdot \mathrm{M} \cdot \mathbf{h} \\
&= \gamma^{\mathrm{t}} \mathbf{h}_{,\mathrm{t}} + \gamma^{\mathrm{r}} \left(\frac{1}{\mathrm{r}} \mathbf{h} + \mathbf{h}_{,\mathrm{r}} \right) + \gamma^{\varphi} \partial_{\varphi} \mathbf{h} + \gamma^{\mathrm{z}} \partial_{\mathrm{z}} \mathbf{h} \pm \mathrm{i} \cdot \mathrm{M} \cdot \mathbf{h} \\
&= \begin{pmatrix}
\frac{\mathrm{i}}{\mathrm{c}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \mathbf{h}_{,\mathrm{t}} + \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \left(\frac{1}{\mathrm{r}} \mathbf{h} + \mathbf{h}_{,\mathrm{r}} \right) \\
+ \frac{\mathrm{i}}{\mathrm{r}} \begin{pmatrix} & & & -1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix} \mathbf{h}_{,\varphi} + \begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix} \mathbf{h}_{,\mathrm{z}} \pm \mathrm{i} \cdot \mathrm{M} \cdot \mathbf{h}
\end{pmatrix} \\
&= \frac{\mathrm{i}}{\mathrm{c}} \begin{pmatrix} \mathrm{h}_{1,\mathrm{t}} \\ \mathrm{h}_{2,\mathrm{t}} \\ -\mathrm{h}_{3,\mathrm{t}} \\ -\mathrm{h}_{4,\mathrm{t}} \end{pmatrix} + \begin{pmatrix} \mathrm{h}_{4,\mathrm{r}} + \frac{\mathrm{h}_4}{\mathrm{r}} - \frac{\mathrm{i}}{\mathrm{r}} \mathrm{h}_{4,\varphi} \\ \mathrm{h}_{3,\mathrm{r}} + \frac{\mathrm{h}_3}{\mathrm{r}} + \frac{\mathrm{i}}{\mathrm{r}} \mathrm{h}_{3,\varphi} \\ -\mathrm{h}_{2,\mathrm{r}} - \frac{\mathrm{h}_2}{\mathrm{r}} + \frac{\mathrm{i}}{\mathrm{r}} \mathrm{h}_{2,\varphi} \\ -\mathrm{h}_{1,\mathrm{r}} - \frac{\mathrm{h}_1}{\mathrm{r}} - \frac{\mathrm{i}}{\mathrm{r}} \mathrm{h}_{1,\varphi} \end{pmatrix} + \begin{pmatrix} \mathrm{h}_{3,\mathrm{z}} \\ -\mathrm{h}_{4,\mathrm{z}} \\ -\mathrm{h}_{1,\mathrm{z}} \\ \mathrm{h}_{2,\mathrm{z}} \end{pmatrix} \pm \mathrm{i} \cdot \mathrm{M} \cdot \begin{pmatrix} \mathrm{h}_1 \\ \mathrm{h}_2 \\ \mathrm{h}_3 \\ \mathrm{h}_4 \end{pmatrix}
\end{aligned} \quad . \tag{68}
$$

Now, a time-radius-dependency is possible and looks as follows:

$$
\begin{aligned}
0 &= \gamma^0 \mathbf{h}_{,0} + \gamma^1 \partial_1 \mathbf{h} + \gamma^2 \partial_2 \mathbf{h} + \gamma^3 \partial_3 \mathbf{h} \pm \mathrm{i} \cdot \mathrm{M} \cdot \mathbf{h} \\
&= \gamma^{\mathrm{t}} \mathbf{h}_{,\mathrm{t}} + \gamma^{\mathrm{r}} \left(\frac{1}{\mathrm{r}} \mathbf{h} + \mathbf{h}_{,\mathrm{r}} \right) + \overbrace{\gamma^{\varphi} \partial_{\varphi} \mathbf{h} + \gamma^{\mathrm{z}} \partial_{\mathrm{z}} \mathbf{h}}^{=0} \pm \mathrm{i} \cdot \mathrm{M} \cdot \mathbf{h} \\
&= \begin{pmatrix}
\frac{\mathrm{i}}{\mathrm{c}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \mathbf{h}_{,\mathrm{t}} + \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \left(\frac{1}{\mathrm{r}} \mathbf{h} + \mathbf{h}_{,\mathrm{r}} \right) \pm \mathrm{i} \cdot \mathrm{M} \cdot \mathbf{h}
\end{pmatrix} \\
&= \frac{\mathrm{i}}{\mathrm{c}} \begin{pmatrix} \mathrm{h}_{1,\mathrm{t}} \\ \mathrm{h}_{2,\mathrm{t}} \\ -\mathrm{h}_{3,\mathrm{t}} \\ -\mathrm{h}_{4,\mathrm{t}} \end{pmatrix} + \begin{pmatrix} \mathrm{h}_{4,\mathrm{r}} + \frac{\mathrm{h}_4}{\mathrm{r}} \\ \mathrm{h}_{3,\mathrm{r}} + \frac{\mathrm{h}_3}{\mathrm{r}} \\ -\mathrm{h}_{2,\mathrm{r}} - \frac{\mathrm{h}_2}{\mathrm{r}} \\ -\mathrm{h}_{1,\mathrm{r}} - \frac{\mathrm{h}_1}{\mathrm{r}} \end{pmatrix} \pm \mathrm{i} \cdot \mathrm{M} \cdot \begin{pmatrix} \mathrm{h}_1 \\ \mathrm{h}_2 \\ \mathrm{h}_3 \\ \mathrm{h}_4 \end{pmatrix}
\end{aligned} \quad . \tag{69}
$$

But how can it be that, while the original Laplace equation (63) is not half that picky with the coordinate dependency of h[…], because there we could have solutions to all sorts of dependencies (67) (see also the full equation in spherical coordinates (70)), meaning solutions can be found to h[t],

h[r], h[φ], h[z], h[t, r], h[t, φ], h[t, z], h[r, φ], h[r, z], h[φ, z], h[t, r, φ], h[t, φ, z], h[t, r, z], h[r, φ, z], the subsequent Dirac equation requires a much more selective approach for certain symmetries?

Well, mathematically it is clear why this is. In the original Laplace equation we have the differential terms structured in such a way that the moment the dependency to one coordinate disappears, the term vanishes. In the Dirac form we find non-derivative terms, caused by the operational factorization, and those create the problems.

For completeness, we here also give the full equation in physical units for the spherical case together with the parameters speed of light in vacuum c, reduced Planck constant $\hbar$, and the ordinary mass m, reading[4]:

$$0=\begin{pmatrix} \frac{i\hbar}{c}\partial_t & 0 & \frac{i\hbar}{r\sin\vartheta}\partial_\varphi & i\hbar\left(-\frac{i}{r}\Theta+R\right) \\ 0 & \frac{i\hbar}{c}\partial_t & i\hbar\left(\frac{i}{r}\Theta+R\right) & -\frac{i\hbar}{r\sin\vartheta}\partial_\varphi \\ -\frac{i\hbar}{r\sin\vartheta}\partial_\varphi & i\hbar\left(\frac{i}{r}\Theta-R\right) & -\frac{i\hbar}{c}\partial_t & 0 \\ i\hbar\left(-\frac{i}{r}\Theta-R\right) & \frac{i\hbar}{r\sin\vartheta}\partial_\varphi & 0 & -\frac{i\hbar}{c}\partial_t \end{pmatrix} \times \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} - mc \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix} \quad . \tag{70}$$

$$\Theta=\frac{1}{\sqrt{\sin\vartheta}}\partial_\vartheta\left(\sqrt{\sin\vartheta}(\ldots)\right);\quad R=\frac{1}{r}\partial_r\left(r(\ldots)\right)$$

[4] E.g., see: Dietterich in https://www.youtube.com/watch?v=gwpWIYBJmq4 or the Warwick-documents at https://warwick.ac.uk/fac/sci/physics/staff/academic/boyd/stuff/dirac.pdf

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