Stable Islands of Dimensionality

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Abstract

In previous papers we have shown that the variation of a system with respect to its dimensionality does not only give us thermodynamics [A1 – A5], and via the Bekenstein-Hawking thought experiment [A6 – A8] a dimensional understanding of the inner structure of black holes [A1 – A5, A9], but even helps us solving the problem of the origin of the fine structure constant [A10] and provides a metric understanding of the Pauli exclusion principle [A11, A12].

Here now we want to investigate the possibility for variational and non-variational conditions for the existence of stable islands of dimensionality in which certain systems can or want to exist in.

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Motivator I: Variation With Respect to Dimensionality?

How Some Dimensions Need Space

Motivated by our desire to fundamentally derive rather basic concepts like, for instance, the Pauli exclusion [1] or the growth of black holes, it was shown in [2] that for spaces of n-spherical symmetry the radius increase with increasing dimension follows the bit-wise growth of a black hole when always demanding the radius for the n-sphere to couple at its extremal dimensional condition. This means that the number of dimensions for the very n-sphere is chosen such that either surface or volume are extremal. The corresponding radius shows the Bekenstein-Hawking behavior [3, 4]:

$$r_f = 2 \cdot L \cdot \sqrt{\pi \cdot \left(n + \sqrt{n \cdot (1+n)} \right)}. \tag{1}$$

Here n gives the dimension, while L is just a scaling factor.

While the classical evaluation is given in the appendix of this paper, we here want to point out a small flaw or inconsistency within the classical derivation and intent to correct it. The basic assumption in Bekenstein's experiment is the construction of a bit-like information, thrown into a black hole by choosing the size of a photon (its wavelength) equal to the Schwarzschild radius. Repeating the evaluation with an uncertainty to this assumption leads to quite some consequences and will later become important within this paper. Thereby the derivation of this refined equation is performed as follows:

At first, following Bekenstein with a slight adjustment, we start with the assumption that the photon's right size should be a wavelength λ of the Schwarzschild radius r_s times an yet unknown parameter μ. Knowing that the energy of the photon would be $E=h*\nu$, with ν denoting the frequency and h giving the Planck constant, and plugging in the equation for the Schwarzschild radius of the photon related mass change Δm (with reduced Planck constant $\hbar$ and the Newton constant G):

$$\begin{gathered} \frac{\Delta r_s \cdot c^4}{2G} = \Delta m \cdot c^2 = \leftarrow \boxed{E = h \cdot \nu} \rightarrow = \frac{h \cdot c}{\lambda} = \frac{h \cdot c}{\mu \cdot r_s} \\ \Rightarrow \frac{\Delta r_s \cdot c^4}{2G} = \frac{h \cdot c}{\mu \cdot r_s} \Rightarrow \Delta r_s \cdot r_s = 2\frac{h \cdot G}{\mu \cdot c^3} = 4\pi \frac{\hbar \cdot G}{\mu \cdot c^3} = 4\pi \cdot \frac{\ell_P^2}{\mu} \Rightarrow \Delta r_s = 4\pi \cdot \frac{\ell_P^2}{\mu \cdot r_s} \end{gathered}, \tag{2}$$

we can derive the surface change of the black hole ΔA as follows:

$$\begin{aligned} \Delta A &= 4\pi\left((\Delta r_s + r_s)^2 - r_s^2 \right) = 4\pi\left(2\Delta r_s \cdot r_s + (\Delta r_s)^2 \right) \\ &= 32 \cdot \pi^2 \cdot \frac{\ell_P^2}{\mu} + 64 \cdot \pi^3 \cdot \frac{\ell_P^4}{\mu \cdot r_s^2} \end{aligned}. \tag{3}$$

Now we assume that we construct a whole black hole just bit by bit and that the latter in the end consists of q bits leading to the identity:

$$q \cdot \Delta A = 4 \cdot \pi \cdot r_s^2 = q \cdot \left(32 \cdot \pi^2 \cdot \frac{\ell_P^2}{\mu} + 64 \cdot \pi^3 \cdot \frac{\ell_P^4}{\mu \cdot r_s^2} \right). \tag{4}$$

Solving with respect to the Schwarzschild radius measured in units of the Planck length, results in:

$$\frac{r_s}{\ell_P} = 2 \cdot \sqrt{\frac{\pi}{\mu} \cdot \left(q + \sqrt{q \cdot (1+q)} \right)}. \tag{5}$$

n-Extremal Objects/Systems

The n-Extremal Sphere

We now consider n-spheres with n denoting the number of dimensions of the sphere and an n-dependent radius and start with the volume function for these objects, reading:

$$V=\frac{\pi^{\frac{n}{2}}}{\Gamma\left[\frac{(n+2)}{2}\right]}\cdot r[n]^{n}. \tag{6}$$

Derivation with respect to n leads to:

$$\frac{\partial V}{\partial n}=\frac{\pi^{\frac{n}{2}}}{n\cdot\Gamma\left[\frac{n}{2}\right]}\cdot r[n]^{n-1}\left(\left(\gamma-H_n\left[\frac{n}{2}\right]+\ln[\pi]+2\cdot\ln\left[r[n]\right]\right)r[n]+2\cdot n\cdot r'[n]\right). \tag{7}$$

Demanding the volume to be extremal, we now obtain the following differential equation of first order in the dimensions n for the radius r[n]:

$$\frac{\partial V}{\partial n}=0=\left(\gamma-H_n\left[\frac{n}{2}\right]+\ln[\pi]+2\cdot\ln\left[r[n]\right]\right)r[n]+2\cdot n\cdot r'[n], \tag{8}$$

resulting in the following solution for the radius r[n]:

$$r[n]=\frac{e^{\frac{C_n}{2n}+\frac{\ln\left[\Gamma\left[1+\frac{n}{2}\right]\right]}{n}}}{\sqrt{\pi}}. \tag{9}$$

Setting the constant C_n zero, we note that for n=0 the radius is not zero, but:

$$r[0]=e^{\frac{1}{2}(-\gamma-\ln[\pi])}=\frac{e^{-\frac{\gamma}{2}}}{\sqrt{\pi}}=0.422751. \tag{10}$$

Surprisingly, leaving the constant finite (and positive), we would even obtain r[0]=∞.

The n-p-Extremal Ellipsoid

Now we generalize our sphere to an n-p-spheroid with p axes being different from r, leading to the following volume formula:

$$V=\frac{\pi^{\frac{n}{2}}}{\Gamma\left[\frac{(n+2)}{2}\right]}\cdot\prod_{j=1}^{n}a_j=\frac{\pi^{\frac{n}{2}}}{\Gamma\left[\frac{(n+2)}{2}\right]}\cdot r^{n}\cdot\beta^{p}[n]. \tag{11}$$

The difference of the two sorts of axes is defined by a factor β. The result for the derivative with respect to the number of dimensions n reads:

$$\frac{\partial V}{\partial n}=\frac{\pi^{\frac{n}{2}}}{n\cdot\Gamma\left[\frac{n}{2}\right]}\cdot r^{n}\beta[n]^{p-1}\left(\left(\gamma-H_n\left[\frac{n}{2}\right]+\ln[\pi]+2\cdot\ln[r]\right)\beta[n]+2p\cdot\beta'[n]\right), \tag{12}$$

$$\frac{\partial \mathrm{V}}{\partial \mathrm{n}}=0=\left(\gamma-\mathrm{H}_{\mathrm{n}}\left[\frac{\mathrm{n}}{2}\right]+\ln[\pi]+2\cdot\ln[\mathrm{r}]\right)\beta[\mathrm{n}]+2\mathrm{p}\cdot\beta'[\mathrm{n}], \tag{13}$$

and gives the following solution for the radius factor β[n]:

$$\beta[\mathrm{n}]=\frac{C_{\mathrm{n}}}{\mathrm{r}^{\frac{n}{\mathrm{p}}}\cdot\pi^{\frac{n}{2\cdot\mathrm{p}}}}\cdot \mathrm{e}^{\frac{\ln\left[\Gamma\left[1+\frac{n}{2}\right]\right]}{\mathrm{p}}}. \tag{14}$$

Other Geometries

We realize that our n-extremal objects are the results of geometry dependent solutions of the volume integral:

$$\begin{aligned}&\delta_{?}\mathrm{W}=0=\delta_{?}\int_{\mathrm{V}}\mathrm{d}^{\mathrm{n}}\mathrm{x}\left(\sqrt{-\mathrm{g}}\cdot\Phi_{\mathrm{R}}[\mathrm{R}]\right)\\&\Rightarrow\delta_{\mathrm{n}}\mathrm{W}=0=\delta_{\mathrm{n}}\int_{\mathrm{V}}\mathrm{d}^{\mathrm{n}}\mathrm{x}\left(\sqrt{-\mathrm{g}}\cdot\left[\Phi_{\mathrm{R}}[\mathrm{R}]=1\right]\right)=\delta_{\mathrm{n}}\int_{\mathrm{V}}\mathrm{d}^{\mathrm{n}}\mathrm{x}\sqrt{-\mathrm{g}},\end{aligned} \tag{15}$$

being variated with respect to its dimensionality. We have already discussed this in a variety of previous publications. For convenience we are going to repeat the essential in the appendix of this paper. In this sub-section we are only interested in a generalization of the cases we have considered so far.

The simplest generalization of (15) (or (87) in the appendix) can be given for an ensemble of N tori of dimensions n_j for the sub-n_j-spheres of the individual torus. The volume integral would then yield:

$$\mathrm{W}=\int_{\mathrm{V}}\mathrm{d}^{\mathrm{n}}\mathrm{x}\left(\sqrt{-\mathrm{g}}\right)=\prod_{\mathrm{j=1}}^{\mathrm{n}}\mathrm{V}_{\mathrm{j}}\left(\mathrm{n}_{\mathrm{j}},\mathrm{r}_{\mathrm{j}}\right)=\prod_{\mathrm{j=1}}^{\mathrm{n}}\mathrm{T}\left[\mathrm{n}_{\mathrm{j}}\right]\cdot\frac{\pi^{\frac{n_{\mathrm{j}}}{2}}}{\Gamma\left[\frac{\left(n_{\mathrm{j}}+2\right)}{2}\right]}\cdot\left(r_{\mathrm{j}}\right)^{n_{\mathrm{j}}}. \tag{16}$$

This could be further generalized for a sum of tori and leaves us with a great variety of pure volume (radii) and dimension variations.

As tori can be seen as combined n_{ij}-spheres with n_{ij} giving the dimension of the sub-spheres constructing, each torus (15) can be generalized as follows when assuming N tori with dimensions n_i and sub-spheres of dimensions n_{ij}:

$$\mathrm{W}=\sum_{\mathrm{i=1}}^{\mathrm{N}}\mathrm{W}_{\mathrm{i}}=\sum_{\mathrm{i=1}}^{\mathrm{N}}\int_{\mathrm{V}_{\mathrm{i}}}\mathrm{d}^{\mathrm{n}_{\mathrm{i}}}\mathrm{x}\left(\sqrt{-\mathrm{g}}\right)=\sum_{\mathrm{i=1}}^{\mathrm{N}}\prod_{\mathrm{j=1}}^{\mathrm{n}_{\mathrm{ij}}}\mathrm{T}\left[\mathrm{n}_{\mathrm{ij}}\right]\cdot\frac{\pi^{\frac{\left(n_{\mathrm{ij}}\right)}{2}}}{\Gamma\left[\frac{\left(n_{\mathrm{ij}}+2\right)}{2}\right]}\cdot\left(r_{\mathrm{ij}}\right)^{n_{\mathrm{ij}}}. \tag{17}$$

Assuming that also complex symmetries of systems could be constructed out of sums of tori, we realize that the variational options are manifold:

$$\delta\mathrm{W}=0=\delta\left(n_{\mathrm{ij}},\mathrm{r}_{\mathrm{ij}}\right)\sum_{\mathrm{i=1}}^{\mathrm{N}}\prod_{\mathrm{j=1}}^{\mathrm{n}_{\mathrm{ij}}}\mathrm{T}\left[\mathrm{n}_{\mathrm{ij}}\right]\cdot\frac{\pi^{\frac{\left(n_{\mathrm{ij}}\right)}{2}}}{\Gamma\left[\frac{\left(n_{\mathrm{ij}}+2\right)}{2}\right]}\cdot\left(r_{\mathrm{ij}}\right)^{n_{\mathrm{ij}}} \tag{18}$$

and leave us with a great variety of options for an optimum sized system in the case of complex symmetries.

We see that even a fairly simple case of n-tori allows for rather complex sets of dimensional extrema.

The Relation Between the Bekenstein- and the n-Sphere Picture

Now, we divide equation (5) by (9) and square the quotient, resulting in:

$$\left(\frac{\mathrm{r_s}}{\mathrm{r}[\mathrm{n}]\cdot\ell_{\mathrm{P}}}\right)^2=\frac{4\cdot\pi^2\cdot\left(q+\sqrt{q\cdot(1+q)}\right)}{\mu\cdot \mathrm{e}^{\left(\frac{C_n}{n}+2\frac{\ln\left[\Gamma\left[1+\frac{n}{2}\right]\right]}{n}\right)}}. \tag{19}$$

In the limiting case q→∞ we can give an accurate solution to the quotient above, reading:

$$\lim_{q=n\to\infty}\left(\frac{\mathrm{r_s}}{\mathrm{r}[\mathrm{n}]\cdot\ell_{\mathrm{P}}}\right)^2=\lim_{q=n\to\infty}\frac{4\cdot\pi^2\cdot\left(q+\sqrt{q\cdot(1+q)}\right)}{\mu\cdot \mathrm{e}^{\left(\frac{C_n}{n}+2\frac{\ln\left[\Gamma\left[1+\frac{n}{2}\right]\right]}{n}\right)}}=\frac{16\cdot\pi^2}{\mu}\cdot\mathrm{e}. \tag{20}$$

In the figures (figures 1 and 2) below we illustrate the n-dependency of the ratio for various constants C_n. We realize that – just as also demonstrated in the appendix - for bigger numbers of dimensionality n, with a suitable fit of the parameters μ and C_n we can always distribute the Bekenstein-Hawking finding (5) by an approach of n-extremal spheres (9).

We therefore conclude that dimensionality plays an important role in the fundamental laws of nature and that dimensionality also is somehow connected with the information (number of bits) a system contains . From there we might also conclude that there are islands of stability in the dimensionality of systems.

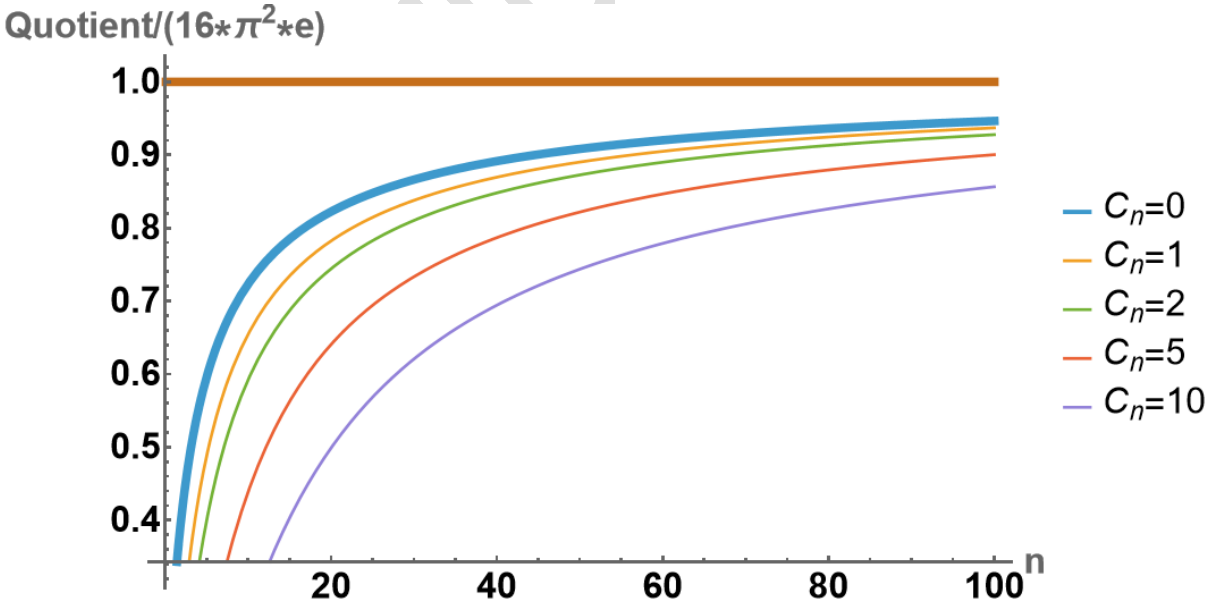


Fig. 1: Quotient (19) for C_n=0 and various positive constants C_n.

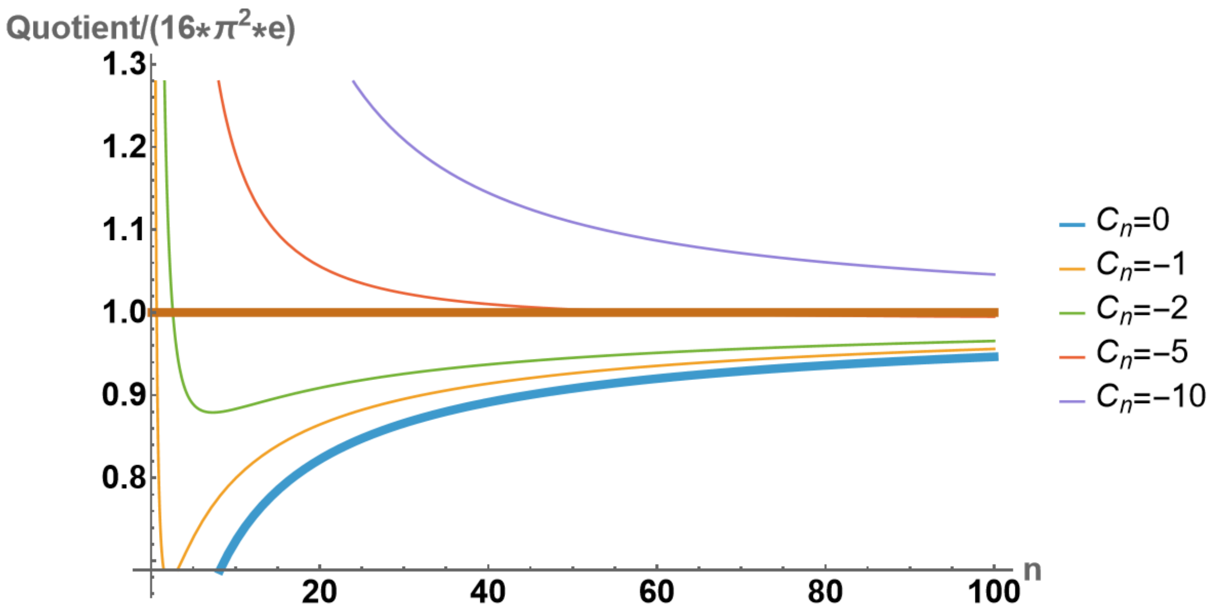


Fig. 2: Quotient (19) for C_n=0 and various negative constants C_n.

Meanwhile we were able to apply the n-variation to a great variety of cases [5 – 12], where we connected not only fundamental quantum equations [13] and solutions to the Einstein-Field-Equations [14] to the dimensional variation, but also considered quite general questions in other fields, like informatic and the derivation of fundamental constants [2, 6, 7, 8, 15 – 21].

The Strange Connection to the Sommerfeld Constant α

Naturally, we may also assume that the number of the fine structure constant just comes out of the quotient in the limit of (20), but instead of n moving towards infinity, we just have a certain dimensionality and need to solve the following equation:

$$\left(\frac{r_s}{r[n]\cdot\ell_P}\right)^2=\frac{4\cdot\pi^2\cdot\left(q+\sqrt{q\cdot(1+q)}\right)}{\mu\cdot e^{\frac{C_n}{n}+2\frac{\ln\left[\Gamma\left[1+\frac{n}{2}\right]\right]}{n}}}=137.035999177=\frac{1}{\alpha}. \tag{21}$$

Thereby we might assume non-black-hole objects and also consider the fact that one bit does not only need one dimensions to be coded but k dimensions. In this case (21) changes to:

$$\left(\frac{r_s}{r[n]\cdot\ell_P}\right)^2=\frac{4\cdot\pi^2\cdot\left(\mathrm{q}+\sqrt{q\cdot(1+q)}\right)}{\mu\cdot e^{\frac{C_n}{k^*q}+2\frac{\ln\left[\Gamma\left[1+\frac{k^*q}{2}\right]\right]}{k^*q}}}=137.035999177=\frac{1}{\alpha}. \tag{22}$$

Setting μ=1, we find that reasonable results can only be expected for k≤3 and the C_n chosen such that we always end up in positive integer n. It should be noted that while C_n plays no role in the limit of n→∞ we, of course, have this as an important parameter in all cases where n is finite. Depending on μ and C_n we than obtain certain n for given systems as “stable islands of dimensionality”.

This aspect will be discussed in [20, 21].

Motivator II: The Classical Equations Already Contained Dimensionality

It needs to be pointed out that the variational equation (15) does not directly relate to the classical approach [27]. Therefore, we intend to work out the connection in this section.

The famous German mathematician David Hilbert [27], even though applying his technique only to derive the Einstein field equations for the General Theory of Relativity [28] in four dimensions, – in principle – extended the classical Hamilton principle to an arbitrary Riemann space-time with a very general variation by not only – as Hamilton and others had done – concentrating on the evolution of the given problem or system in time, but with respect to all its dimensions. His formulation of the Hamilton extremal principle looked as follows:

$$\delta W = 0 = \delta \int_V d^n x \left(\sqrt{-g} \cdot \left(R - 2\Lambda + L_M \right) \right). \tag{23}$$

There, we have the Ricci scalar of curvature R, the cosmological constant Λ, the Lagrange density of matter L_M, and the determinant g of the metric tensor of the Riemann space-time $g_{\alpha\beta}$. For historical reasons, it should be mentioned that Hilbert's original work [27] did not contain the cosmological constant, because it was added later by Einstein in order to obtain a static universe, but this is not of any importance here. The evaluation of the so-called Einstein-Hilbert action (23) brought indeed Einstein's General Theory of Relativity [28], but it did not produce the other great theory physicists have found, which is Quantum Theory. It was not before Schwarzer, about one hundred years after the publication of Hilbert's paper [27], extended Hilbert's approach by considering scaling factors to the metric tensor and showed that Quantum Theory already resides inside the sufficiently general General Theory of Relativity [2, 5, 6, 7, 8]. We will not discuss the reason why this simple idea has not been tried out by other scientists before, but we may still express our amazement about the fact that a simple extension of the type:

$$G_{\alpha\beta} = g_{\alpha\beta} \cdot F[f] \tag{24}$$

solves one of the greatest problems in science, namely the unification of physics and that it took science more than 100 years to come up with the idea. Using the symbol G for the determinant of the scaled metric tensor $G_{\alpha\beta}$ from (24) of the Riemann space-time, we can rewrite the Einstein-Hilbert action from (23) as follows:

$$\delta W = 0 = \delta \int_V d^n x \left(\sqrt{-G} \cdot F^q \cdot \left(R^* - 2\Lambda + L_M \right) \right). \tag{25}$$

Variation is also possible and still converges to the classical form for F→1. Here, which is to say in this paper, we will mainly consider examples with q=0, but for completeness and later investigation we shall mention that a comprehensive consideration of variational integrals for the cases of general q are to be found in [8].

Performing the variation in (25) with respect to the metric $G_{\alpha\beta}$ and remembering that the Ricci curvature of such a scaled metric contains certain derivatives of F (see some previous publications, e.g., [6] appendix D), changes the whole variation to:

$$
\begin{aligned}
\delta W = 0 &= \delta\int_V d^n x\left(\sqrt{-G}\cdot F^q\cdot\left(R^* - 2\Lambda + L_M\right)\right) \\
&= \delta\int_V d^n x\left(\sqrt{-G}\cdot F^q\cdot\left(\left(\frac{R}{F} - \frac{1}{2F^2}\left(\begin{matrix}(n-1)\left(\overbrace{2g^{ab}F_{,ab} + F_{,d}g^{cd}g^{ab}g_{ab,c}}^{=2\Delta F - 2F_{,d}g^{cd}{}_{,c}}\right) \\ -nF_{,d}g^{cd}g^{ab}g_{ac,b}\end{matrix}\right) - (n-1)\frac{g^{ab}F_{,a}\cdot F_{,b}}{4F^3}(n-6)\right) - 2\Lambda + L_M\right)\right),
\end{aligned}
\tag{26}
$$

and results in:

$$
\begin{aligned}
0 &= \left(R^*{}_{\alpha\beta} - \frac{1}{2}R^*\cdot G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta} + g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}} \\
&= \left(\begin{matrix}\left(R_{\alpha\beta} - \frac{1}{2F}\left(\begin{matrix}F_{,\alpha\beta}(n-2) + F_{,ab}g_{\alpha\beta}g^{ab} + F_{,a}g^{ab}\left(g_{\beta b,\alpha} - g_{\beta\alpha,b}\right) - \\ F_{,\alpha}g^{ab}g_{\beta b,a} - F_{,\beta}g^{ab}g_{\alpha b,a} + F_{,d}g^{cd}\left(\begin{matrix}g_{\alpha c,\beta} - \frac{1}{2}ng_{\alpha c,\beta} - \frac{1}{2}ng_{\beta c,\alpha} \\ + \frac{1}{2}ng_{\alpha\beta,c} + \frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{matrix}\right)\end{matrix}\right) + \frac{1}{4F^2}\left(F_{,\alpha}\cdot F_{,\beta}(3n-6) + g_{\alpha\beta}F_{,c}F_{,d}g^{cd}(4-n)\right)\right)\delta G^{\alpha\beta} \\ +(n-1)\left(\frac{1}{2F}\left(\begin{matrix}\left(\overbrace{2g^{ab}F_{,ab} + F_{,d}g^{cd}g^{ab}g_{ab,c}}^{=2\Delta F - 2F_{,d}g^{cd}{}_{,c}}\right) \\ - \frac{n}{(n-1)}F_{,d}g^{cd}g^{ab}g_{ac,b}\end{matrix}\right) + \frac{g^{ab}F_{,a}\cdot F_{,b}}{4F^2}(n-6) - \frac{R}{(n-1)}\right)\cdot\frac{g_{\alpha\beta}}{2}\end{matrix}\right),
\end{aligned}
\tag{27}
$$

when setting q=0 and assuming a vanishing cosmological constant. With a cosmological constant we have to write:

$$0=\left(R^{*}{}_{\alpha\beta}-\frac{1}{2}R^{*}\cdot G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}$$

$$=\left(\begin{array}{c}\boxed{\boxed{R_{\alpha\beta}-R\frac{g_{\alpha\beta}}{2}}}+\boxed{\Lambda\cdot g_{\alpha\beta}}\\ -\frac{1}{2F}\left(\begin{array}{c}F_{,\alpha\beta}(n-2)+F_{,ab}g_{\alpha\beta}g^{ab}+F_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-\\ F_{,\alpha}g^{ab}g_{\beta b,a}-F_{,\beta}g^{ab}g_{\alpha b,a}+F_{,d}g^{cd}\left(\begin{array}{c}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)\end{array}\right)\\ +\frac{1}{4F^{2}}\left(F_{,\alpha}\cdot F_{,\beta}(3n-6)+g_{\alpha\beta}F_{,c}F_{,d}g^{cd}(4-n)\right)\\ +(n-1)\left(\frac{1}{2F}\left(\begin{array}{c}2\Delta F-2F_{,d}g^{cd}{}_{,c}\\ -\frac{n}{(n-1)}F_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right)+\frac{g^{ab}F_{,a}\cdot F_{,b}}{4F^{2}}(n-6)\right)\cdot\frac{g_{\alpha\beta}}{2}\end{array}\right)\delta G^{\alpha\beta}\quad. \qquad (28)$$

For better recognition of the classical terms, we have reordered a bit and boxed the classical vacuum part of the Einstein field equations (double lines) and the cosmological constant term (single line). Everything else can be – no, represents (!) – matter or quantum effects or both.

Thus, we also – quite boldly – have set the matter density L_M equal to zero, because we see that already our simple metric scaling brings in quite some options for the construction of matter. We have seen in our previous work [2, 5, 6, 7, 8] that there is much more with the same technique (see especially [5]). We also discussed more general kernels for the Hilbert action and explained why this does not necessarily lead to deviations from the gravitational laws we observe [5] even though the rigorous performance of the variation for such kernels would suggest such a deviation [29]… on first glance.

What we also realize is that we already find the variational ingredients for the n-extremal systems, we have considered above. We simple take the classical equation (23) or its generalized form (25) (with q=0) and generalize the variation with respect to all aspects of the metric, which automatically also includes its dimensionality. We might want to write it like:

$$\delta W=0=\delta_{n}\left[\delta_{G_{\alpha\beta}}\int_{V}d^{n}x\left(\sqrt{-G}\cdot\left(R^{*}-2\Lambda+L_{M}\right)\right)\right]. \qquad (29)$$

We see that the case with no curvature yields:

$$\delta W=0=\delta_{n}\left[\int_{V}d^{n}x\left(\sqrt{-G}\cdot\left(T_{\alpha\beta}-\Lambda\cdot G_{\alpha\beta}\right)\right)\delta G^{\alpha\beta}\right]$$
$$=\delta_{n}\left[\int_{V}d^{n}x\left(\sqrt{-G}\cdot\left(T_{\alpha\beta}-\Lambda\cdot F\cdot g_{\alpha\beta}\right)\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}\right]\quad. \qquad (30)$$
$$=\delta_{n}\left[\int_{V}d^{n}x\left(\sqrt{-G}\cdot\left(T_{\alpha\beta}\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)-\Lambda\cdot g_{\alpha\beta}\delta g^{\alpha\beta}-\Lambda\cdot F\cdot n\cdot\delta\left(\frac{1}{F}\right)\right)\right)\right]$$

Assuming vanishing matter and a small metric variation results in:

$$
\begin{aligned}
0 &= \delta_n\left[\int_V d^n x\left(\sqrt{-G}\cdot\left(\overbrace{T_{\alpha\beta}\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{=0}-\Lambda\cdot g_{\alpha\beta}\,\overbrace{\delta g^{\alpha\beta}}^{\to 0}-\Lambda\cdot F\cdot n\cdot\delta\left(\frac{1}{F}\right)\right)\right)\right]. \\
&= -\delta_n\left[\int_V d^n x\left(\sqrt{-G}\cdot\left(\Lambda\cdot F\cdot n\cdot\delta\left(\frac{1}{F}\right)\right)\right)\right]
\end{aligned}
\tag{31}
$$

We see that the volume integral, we have used above for our n-extremal objects only appears when we treat the remaining terms in (31) as constants, which is to say:

$$
\begin{aligned}
&\Lambda\cdot F\cdot n\cdot\delta\left(\frac{1}{F}\right)=\text{const}\equiv\tilde{\Lambda} \\
\Rightarrow\quad &0=\delta_n\left[\int_V d^n x\left(\sqrt{-G}\cdot\tilde{\Lambda}\right)\right]=\tilde{\Lambda}\cdot\delta_n\int_V d^n x\sqrt{-G}
\end{aligned}
\tag{32}
$$

This means that when after the usual Hilbert metric variation a constant remains, we find ourselves in the position to perform the n-extremal search just on an ordinary volume integral. Most interestingly, these conditions are fulfilled in the cases of black holes following the Schwarzschild metric [30] as we have considered them in here.

Minimalizing the “Empty” Space-Time

Assuming that any n-dimensional space-time might be filled with geometric objects of a certain symmetry, one automatically has to ask the question what happens in those cases where this symmetry does not allow for a complete filling of the space-time. We just need to imagine spheres in boxes or stacks of cannon balls in order to see “the potential problem”. Knowing that cartesian coordinates always allow a complete filling, we can define the volume V_R of the space being “left out” by an ensemble of objects of a certain symmetry:

$$V_R = V_C - V_O. \tag{33}$$

Here V_C denotes the Cartesian volume and V_O the volume occupied by the objects.

Of course, one could now, as before, ask for an extremum for V_R just being constructed our of extrema of its addends, but here, we want to investigate a non-variational possibility by just demanding:

$$V_R = V_C - V_O = 0. \tag{34}$$

Let us consider a space filled with n-spheres in such a way that each sphere occupies exactly a cube of that very n-space. Naturally, this is not the optimum packing with the highest possible sphere density. Then (34) reads as follows:

$$V_R = V_C - V_O = \left(2\cdot r[n]\right)^n - \frac{\pi^{\frac{n}{2}}}{\Gamma\left[\frac{(n+2)}{2}\right]}\cdot r[n]^n. \tag{35}$$

Keen on making the rest-space to vanish, we find a solution only for n=1, because then we have:

$$0=(2\cdot r[\mathrm{n}])^{\mathrm{n}}-\frac{\pi^{\frac{\mathrm{n}}{2}}}{\Gamma\left[\frac{(n+2)}{2}\right]}\cdot r[\mathrm{n}]^{\mathrm{n}}=\left(2^{\mathrm{n}}-\frac{\pi^{\frac{\mathrm{n}}{2}}}{\Gamma\left[\frac{(n+2)}{2}\right]}\right)\cdot r[\mathrm{n}]^{\mathrm{n}}, \tag{36}$$

but when assuming that there might be structures where, instead of the factor 2 for the n-cube's volume with the cube-centered n-sphere:

$$\mathrm{V_C}=(2\cdot r[\mathrm{n}])^{\mathrm{n}}, \tag{37}$$

we could have something like:

$$\mathrm{V_C}=(\alpha\cdot r[\mathrm{n}])^{\mathrm{n}}, \tag{38}$$

we'd obtain a whole set of solutions for various n in dependence on the factor α. The figure (fig. 3) below illustrates the corresponding distribution.

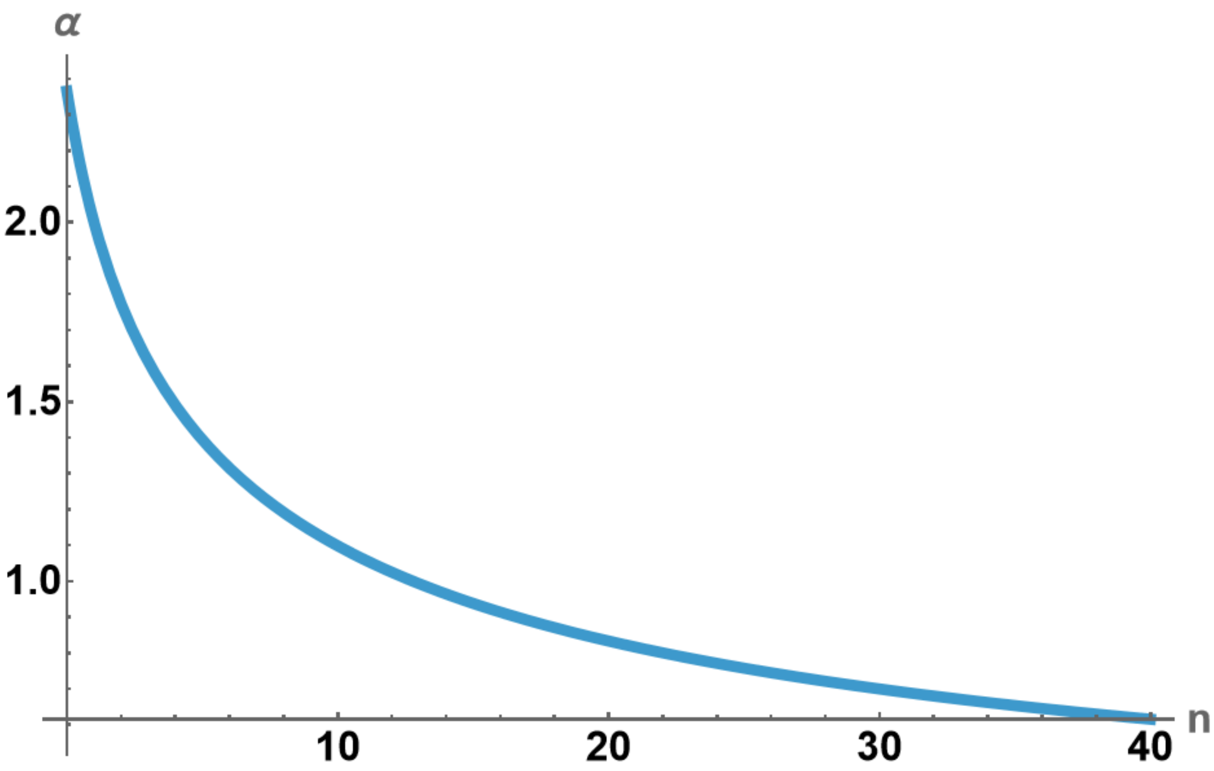


Fig. 3: The factor α in dependence on the dimension n.

Of course, we might also just want to be the rest-space to be extremal, which requires us to solve the following equation:

$$\frac{\partial}{\partial \mathrm{n}}\mathrm{V_R}=\frac{\partial}{\partial \mathrm{n}}(\mathrm{V_C}-\mathrm{V_O})=\frac{\partial}{\partial \mathrm{n}}\left(\alpha^{\mathrm{n}}-\frac{\pi^{\frac{\mathrm{n}}{2}}}{\Gamma\left[\frac{(n+2)}{2}\right]}\right)\cdot r[\mathrm{n}]^{\mathrm{n}}. \tag{39}$$

Unfortunately, we are unable to find a solution here and thus, leave the further consideration of this option to the interested reader.

Thermodynamics of Dimensionality

In a variety of previous publications we have shown that the variation with respect to the number of degrees of freedom gives thermodynamics. As this is a pretty comprehensive topic, the interested reader is referred to our original work [22 - 26].

Collapse of the Wavefunction-“Collapse”

Even though it is not really a topic of this paper, the dimensionality of a system also has an impact on how we observe things. As an observation of anything requires the observer to somehow entangle with the object of his observation, the number of dimensions of the previously separated systems observer and object has to be combined or even changes when being forced together during the observation (measurement).

We are not going to completely investigate this problem in here, but briefly show on the example of the hydrogen atom what “observation” metrically means.

A 7-dimensional Schrödinger Hydrogen Atom

Taking the metric of a spacetime with dimensions t, r, ϑ, φ, φ_1, φ_2 and φ_3 of the following form:

$$g_{\alpha\beta}=f\left[t,r,\vartheta,\varphi,\varphi_3\right]^{\frac{4}{5}}\cdot\begin{pmatrix}-c^2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & r^2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & r^2\cdot\sin^2\vartheta & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & r^{-p-1} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & r^p & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & r\end{pmatrix};\quad p^2+p+1=0$$
$$f\left[t,r,\vartheta,\varphi,\varphi_3\right]=f_{\text{Schrödinger}}\left(r,\vartheta,\varphi\right)\cdot\begin{pmatrix}\left(C_{tc}\cos\left(c\cdot\sqrt{-Q_n}\cdot t\right)+C_{ts}\sin\left(c\cdot\sqrt{-Q_n}\cdot t\right)\right)\\ \times\left(C_{\varphi c}\cos\left(C_{\varphi_3}\cdot\varphi_3\right)+C_{\varphi s}\sin\left(C_{\varphi_3}\cdot\varphi_3\right)\right)\end{pmatrix},\qquad(40)$$

assuming “weak gravity” according to:

$$\delta G^{\alpha\beta}=g^{\alpha\beta}\cdot\delta_0+\overbrace{g^{ab}\delta_{ab}^{\alpha\beta}}^{\text{Gravity}}\xrightarrow{\forall\,\delta_{ab}^{\alpha\beta}\ll\delta_0}=g^{\alpha\beta}\cdot\delta_0,\qquad(41)$$

and feeding everything into the Quantum Einstein-Field-Equations [2, 5, 6, 7, 8]:

$$
0=\left(R^{*}{}_{\alpha\beta}-\frac{1}{2}R^{*}\cdot G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}
$$

$$
=\left(\begin{aligned}
&\overbrace{\left(\begin{aligned}
&R_{\alpha\beta}-\frac{1}{2F}\left(\begin{aligned}
&F_{,\alpha\beta}(n-2)+F_{,ab}g_{\alpha\beta}g^{ab}+F_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-\\
&F_{,\alpha}g^{ab}g_{\beta b,a}-F_{,\beta}g^{ab}g_{\alpha b,a}+F_{,d}g^{cd}\left(\begin{aligned}&g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\&+\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{aligned}\right)
\end{aligned}\right)\\
&+\frac{1}{4F^{2}}\left(F_{,\alpha}\cdot F_{,\beta}(3n-6)+g_{\alpha\beta}F_{,c}F_{,d}g^{cd}(4-n)\right)
\end{aligned}\right)}^{R^{*}{}_{\alpha\beta}}\\
&\underbrace{+(n-1)\left(\frac{1}{2F}\left(\begin{aligned}&\left(\overbrace{2g^{ab}F_{,ab}+F_{,d}g^{cd}g^{ab}g_{ab,c}}^{=2\Delta F-2F_{,d}g^{cd}{}_{,c}}\right)\\&-\frac{n}{(n-1)}F_{,d}g^{cd}g^{ab}g_{ac,b}\end{aligned}\right)+\frac{g^{ab}F_{,a}\cdot F_{,b}}{4F^{2}}(n-6)-\frac{R}{(n-1)}\right)\cdot\frac{g_{\alpha\beta}}{2}}_{-\frac{1}{2}R^{*}\cdot G_{\alpha\beta}}
\end{aligned}\right)\delta G^{\alpha\beta}, \quad (42)
$$

being contracted with the contravariant metric tensor in accordance with the condition (41), directly leads us to the following scalar quantum gravity equation for the function $f_{\text{Schrödinger}}(r,\vartheta,\varphi)$:

$$
\left[-\Delta_{\text{Schrödinger}}+\overbrace{\frac{C_{\varphi_3}^{2}}{r}}^{V_{\text{Schrödinger H-atom}}}\right]f_{\text{Schrödinger}}(r,\vartheta,\varphi)=Q_n f_{\text{Schrödinger}}(r,\vartheta,\varphi)\,. \quad (43)
$$

In order to obtain the usual negative 1/r-dependency plus the positive energy in (43) [31, 32] and still having a stable oscillation instead of hyperbolic behavior for the two functions:

$$
f_{[t,\varphi_3]}[t,\varphi_3]=f_t[t]\cdot f_{\varphi_3}[\varphi_3]=\left(\begin{aligned}&\left(C_{tc}\cos\left(c\cdot\sqrt{Q_n}\cdot t\right)+C_{ts}\sin\left(c\cdot\sqrt{Q_n}\cdot t\right)\right)\\&\times\left(C_{\varphi c}\cos\left(C_{\varphi_3}\cdot\varphi_3\right)+C_{\varphi s}\sin\left(C_{\varphi_3}\cdot\varphi_3\right)\right)\end{aligned}\right), \quad (44)
$$

however, we see that we have to make the last coordinate time-like and the time coordinate space-like, which is to say (see red squares):

$$g_{\alpha\beta}=f\left[t,r,\vartheta,\varphi,\varphi_3\right]^{\frac{4}{5}}\cdot\begin{pmatrix} \boxed{+c^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2\cdot\sin^2\vartheta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r^{-p-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r^p & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{-r} \end{pmatrix};\quad p^2+p+1=0 \quad . \tag{45}$$

$$f\left[t,r,\vartheta,\varphi,\varphi_3\right]=f_{\text{Schrödinger}}\left(r,\vartheta,\varphi\right)\cdot\begin{pmatrix} \left(C_{tc}\cos\left(c\cdot\sqrt{Q_n}\cdot t\right)+C_{ts}\sin\left(c\cdot\sqrt{Q_n}\cdot t\right)\right) \\ \times\left(C_{\varphi c}\cos\left(C_{\varphi_3}\cdot\varphi_3\right)+C_{\varphi s}\sin\left(C_{\varphi_3}\cdot\varphi_3\right)\right) \end{pmatrix}$$

This allows us for real positive parameters Q_n and C_{φ_3} and would give us the following partial differential equation:

$$\left[-\Delta_{\text{Schrödinger}}-\frac{\overbrace{C_{\varphi_3}^2}^{V_{\text{Schrödinger H-atom}}}}{r}\right]f_{\text{Schrödinger}}\left(r,\vartheta,\varphi\right)+Q_n f_{\text{Schrödinger}}\left(r,\vartheta,\varphi\right)=0 \quad . \tag{46}$$

$$\text{with}:\quad C_{\varphi_3}^2=\frac{q^2\cdot\mu}{2\pi\varepsilon_0\hbar^2};\quad Q_n=E_n\frac{2\cdot\mu}{\hbar^2};\quad \mu=\frac{m_e m_p}{m_e+m_p}$$

With $\hbar$, μ, q and ε_0 denoting the reduced Planck constant, the reduced mass (with mass of proton m_p and electron m_e), the unit charge and the permittivity constant in vacuum, the well-known corresponding solution can be given as follows [31 - 34]:

$$\begin{aligned} f_{\text{Schrödinger}}\left(r,\vartheta,\varphi\right)&=\Psi_{n,l,m}\left[r,\vartheta,\varphi\right]=e^{i\cdot m\cdot\varphi}\cdot P_l^m\left[\cos\vartheta\right]\cdot R_{n,l}\left[r\right] \\ &=\sqrt{\left(\frac{2}{n\cdot a_0}\right)^3\frac{(n-l-1)!}{2\cdot n\cdot(n+l)!}}\cdot e^{-\rho/2}\cdot\rho^l\cdot L_{n-l-1}^{2l+1}\left[\rho\right]\cdot Y_l^m\left[\vartheta,\varphi\right];\quad \rho=\frac{2\cdot r}{n\cdot a_0}\cdot \end{aligned} \tag{47}$$

The constant a_0 is denoting the Bohr radius with:

$$a_0=\frac{4\cdot\pi\cdot\varepsilon_0\cdot\hbar^2}{m_e\cdot q^2}=5.292\cdot10^{-11}\text{meter} . \tag{48}$$

The functions P, L and Y denote the associated Legendre function, the Laguerre polynomials and the spherical harmonics, respectively.

Regarding the classical discussion of the solution, especially the quantization via the quantum numbers n, l and m, we have to refer to the text book literature (e.g. [31, 32]).

Here we are only interested in the influence of the potential- and energy-related other dimensions t, φ_1, φ_2 and φ_3 on the character of the solution (bonded or non-bonded states).

As we require negative E_n for the bonded states, we see that changing the approach (40) to pure oscillations in time as follows:

$$f\left[t,r,\vartheta,\varphi,\varphi_3\right]=f_{\text{Schrödinger}}\left(r,\vartheta,\varphi\right)\cdot\begin{pmatrix}\left(C_{tc}\cos\left(c\cdot\sqrt{Q_n}\cdot t\right)+C_{ts}\sin\left(c\cdot\sqrt{Q_n}\cdot t\right)\right)\\ \times\left(C_{\varphi c}\cos\left(C_{\varphi_3}\cdot\varphi_3\right)+C_{\varphi s}\sin\left(C_{\varphi_3}\cdot\varphi_3\right)\right)\end{pmatrix}, \tag{49}$$

does give us exactly what we need. A hyperbolic approach in time:

$$f\left[t,r,\vartheta,\varphi,\varphi_3\right]=f_{\text{Schrödinger}}\left(r,\vartheta,\varphi\right)\cdot\begin{pmatrix}\left(C_{tc}\cosh\left(c\cdot\sqrt{Q_n}\cdot t\right)+C_{ts}\sinh\left(c\cdot\sqrt{Q_n}\cdot t\right)\right)\\ \times\left(C_{\varphi c}\cos\left(C_{\varphi_3}\cdot\varphi_3\right)+C_{\varphi s}\sin\left(C_{\varphi_3}\cdot\varphi_3\right)\right)\end{pmatrix}, \tag{50}$$

on the other hand, gives us continuous (non-bonded) solutions.

We see that we are able to construct the metric situation for the Schrödinger Hydrogen problem in a 7-dimensional space-time.

Along the way, however, in order to obtain the right signs with respect to the classical Schrödinger equation, we had to change the classical time-coordinate to space-like and the coordinate making up our 1/r-potential (which here was φ_3) to time-like. If we wanted to avoid such changes, we might stick to the metric (40) and consider non-oscillating hyperbolic solutions for t and φ_3 as follows:

$$\begin{aligned}f\left[t,r,\vartheta,\varphi,\varphi_3\right]&=f_{\text{Schrödinger}}\left(r,\vartheta,\varphi\right)\cdot\begin{pmatrix}\left(C_{tc}\cosh\left(c\cdot\sqrt{Q_n}\cdot t\right)+C_{ts}\sinh\left(c\cdot\sqrt{Q_n}\cdot t\right)\right)\\ \times\left(C_{\varphi c}\cosh\left(C_{\varphi_3}\cdot\varphi_3\right)+C_{\varphi s}\sinh\left(C_{\varphi_3}\cdot\varphi_3\right)\right)\end{pmatrix}\\ &=f_{\text{Schrödinger}}\left(r,\vartheta,\varphi\right)\cdot\begin{pmatrix}\left(C_{t-}\cdot e^{-c\cdot\sqrt{Q_n}\cdot t}+C_{t+}\cdot e^{+c\cdot\sqrt{Q_n}\cdot t}\right)\\ \times\left(C_{\varphi-}\cdot e^{-C_{\varphi_3}\cdot\varphi_3}+C_{\varphi+}\cdot e^{+C_{\varphi_3}\cdot\varphi_3}\right)\end{pmatrix}\end{aligned}. \tag{51}$$

In order to obtain stable 1/r-potentials and energies, of course, we have then to assume system coordinates t and φ_3, which are „fixed" around certain values keeping the boundary conditions and thus, maintaining the structure of equation (46). In other words, the t in metric (40) is not to be mixed up with the general omnipresent time we experience in our daily life, but would only be a kind of system time, which would be frozen at a certain value in order not to produce singularities for infinitely growing t. The same has to hold for the potential coordinate φ_3. The system appears static, which is to say, it takes on certain states, meaning it jumps to certain parameters and then just stays there. It would also be possible to assume "moving coordinates t and φ_3" and just live with the fact that there are internally growing hyperbolic dependencies for the functions f_t and f_{φ_3} which simply do not show up during the observation of the system, because, either such changes are too small to be seen, or the system permanently resets itself. We admit, however, that the adjustment of time- and space-like plus the choice of oscillating solutions appears a bit more attractive.

An 8-dimensional Schrödinger Hydrogen Atom

Let us assume a dimensionally small system of only 8 degrees of freedom of the following kind:

$$
g^{8}_{\alpha\beta}=\begin{pmatrix} c^2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & r^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & r^2\cdot\sin^2\varphi_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{77} \end{pmatrix}
$$

$$
\times F\left[f\left[t,r,\vartheta,\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5\right]\right];\quad F[f]=f^{\frac{2}{3}}
$$

$$
f\left[t,r,\vartheta,\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5\right]=f_t[t]\cdot f_r[r]\cdot f_\vartheta[\vartheta]\cdot\prod_{i=1}^{5}f_{\varphi_i}[\varphi_i];
$$

$$
g_{44}=\left(g_{\varphi_2}[\varphi_2]'\right)^2\cdot r^{i\cdot s};
$$

$$
g_{55}=\left(g_{\varphi_3}[\varphi_3]'\right)^2\cdot r^{s};
$$

$$
g_{66}=\left(g_{\varphi_4}[\varphi_4]'\right)^2\cdot r^{-i\cdot s};
$$

$$
g_{77}=-\left(g_{\varphi_5}[\varphi_5]'\right)^2\cdot r^{-s};
\tag{52}
$$

we can get the same partial differential equation as given above. Our next step is to apply a separation approach and to partially "fix" some of the functions as follows (for i=3, 4, 5[1]):

$$
f_{\varphi_i}\left[\varphi_i\right]=f_{\varphi_i}\left[g_{\varphi_i}\left[\varphi_i\right]\right]\equiv f_{\varphi_i}\left[g_{\varphi_i}\right];
$$

$$
f_{\varphi_i}\left[g_{\varphi_i}\right]=C_{-i}\cdot e^{-i\cdot A_i\cdot g_{\varphi_i}}+C_{+i}\cdot e^{+i\cdot A_i\cdot g_{\varphi_i}}=C_{ci}\cdot\cos\left[A_i\cdot g_{\varphi_i}\right]+C_{si}\cdot\sin\left[A_i\cdot g_{\varphi_i}\right]
$$

$$
\text{especially}\quad(\text{classically}):
$$

$$
f_{\varphi 1}\left[g_{\varphi_1}\right]=C_{-1}\cdot e^{-i\cdot A_1\cdot g_{\varphi_1}}+C_{+1}\cdot e^{+i\cdot A_1\cdot g_{\varphi_1}}=C_{c1}\cdot\cos\left[\overbrace{A_1}^{\hat{=}m}\cdot g_{\varphi_1}\right]+C_{s1}\cdot\sin\left[\overbrace{A_1}^{\hat{=}m}\cdot g_{\varphi_1}\right]
\tag{53}
$$

$$
f_t[t]=C_{t1}\cdot\cos[c\cdot E\cdot t]+C_{t2}\cdot\sin[c\cdot E\cdot t],
\tag{54}
$$

$$
f_\vartheta\left[\vartheta=\varphi_0\right]=C_{P\vartheta}\cdot P_L^{A_1}\left[\cos\left[\vartheta=\varphi_0\right]\right]+C_{Q\vartheta}\cdot Q_L^{A_1}\left[\cos\left[\vartheta=\varphi_0\right]\right],
\tag{55}
$$

with the associated Legendre polynomials $P_L^{A_1},Q_L^{A_1}$. This gives us the following differential equation for the r-dependency of f[…]:

[1] If not used as index "i" stands for the imaginary number $i=\sqrt{-1}$.

$$
\begin{gathered}
R^{*}=0=\left(\frac{L\cdot(L+1)}{r^{2}}+r^{i\cdot s}\cdot A_{2}^{2}+r^{s}\cdot A_{3}^{2}+r^{-i\cdot s}\cdot A_{4}^{2}-r^{-s}\cdot A_{5}^{2}-\overbrace{\Delta_{3D-\mathrm{sphere}}[r]}^{2\frac{\partial}{r\cdot\partial r}+\frac{\partial^{2}}{\partial r^{2}}}+E^{2}\right)\Psi \\
\Psi=\Psi[r,\vartheta=\varphi_{0},\varphi=\varphi_{1}]=f_{r}[r]\cdot f_{\vartheta}[\vartheta=\varphi_{0}]\cdot f_{\varphi}[\varphi=\varphi_{1}] \\
\Rightarrow \\
0=f_{\vartheta}[\vartheta]\cdot f[\varphi]\cdot\left(\begin{array}{c}\frac{L\cdot(L+1)}{r^{2}}+r^{i\cdot s}\cdot A_{2}^{2}+r^{s}\cdot A_{3}^{2}+r^{-i\cdot s}\cdot A_{4}^{2}\\ -r^{-s}\cdot A_{5}^{2}-\frac{2\partial}{r\cdot\partial r}-\frac{\partial^{2}}{\partial r^{2}}+E^{2}\end{array}\right)f_{r}[r]
\end{gathered}
\quad . \qquad (56)
$$

We see that for the choice of s=1 and $f_{\varphi_i}[\varphi_i]=\mathrm{const}$ for i=2,3,4, we obtain:

$$
0=f_{\vartheta}[\vartheta]\cdot f[\varphi]\cdot\left(\frac{L\cdot(L+1)}{r^{2}}-\frac{A_{5}^{2}}{r}-\frac{2\partial}{r\cdot\partial r}-\frac{\partial^{2}}{\partial r^{2}}+E^{2}\right)f_{r}[r], \qquad (57)
$$

which is just the classical Schrödinger hydrogen problem with the corresponding solution:

$$
\begin{gathered}
f_{r}[r]=e^{-r\cdot E}\cdot r^{L}\cdot\left(C_{U}U[-nn,2\cdot(1+L),2\cdot r\cdot\mu]+C_{L}\cdot L_{nn}^{1+2\cdot L}[2\cdot r\cdot\mu]\right) \\
nn=\frac{A_{5}^{2}}{2\cdot E}-1-L
\end{gathered}
\quad . \qquad (58)
$$

Bringing in the Observer

No matter which of the metric constellations for the construction of the hydrogen atom we might have used, when we want to find out in what of the possible states the atom is in, we are forced to connect (entangle) our own – huge and many-dimensional – metric (which is or mathematically represents us) with the hydrogen system, thereby forming one big combined space-time. Obviously, the combined metric is not equal to the original hydrogen metrics ((40), (45) or (52)) even though, it might still contain it in an almost undisturbed but entangled manner. The entanglement with the observer makes all the difference. Taking the example of (52) we now have:

$$
\Omega_{\mu\nu}^{"\infty"}=\Omega_{\alpha,\mu|\beta,\nu}^{"\infty"}=\Psi\cdot\begin{pmatrix} g_{\alpha\beta}^{8} & O_{\mu\beta}\\ O_{\alpha\nu} & O_{\mu\nu}\end{pmatrix}, \qquad (59)
$$

with the new wave function Ψ acting as the connector between the observer and the observed.

Of course, Ψ is not F and in the usual macroscopic reality probably has not much to do with the hydrogen atom apart from the fact that it somehow mirrors the connection to the observer during the measurement process. So, if the observer is interested in the information within F, he somehow has to make sure that everything else does not become too dominant. Still, he will never be able to truly just measure F completely on its own, because no matter, how clever the metric setup (59) might look like, the Ψ, the observer observes, will never only contain just information related to F and the metric (52) but always also a bit of the “rest”, which means the observer and his macroscopic reality. The observer may get close to a “pure” measurement of F by trying to restrict the dependencies of Ψ to the coordinates of (52). The quantum gravitational metric to be run through the variation process does then read:

$$\Omega^{n}_{\mu\nu}=\Psi\left[\psi\left[t,r,\vartheta,\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5\right]\right]\cdot\begin{pmatrix} g^{8}_{\alpha\beta} & O_{\mu\beta} \\ O_{\alpha\nu} & O_{\mu\nu}\end{pmatrix}\equiv\Psi\left[\psi\right]\cdot\omega_{\mu\nu}\,. \tag{60}$$

From the Quantum Einstein Field Equations (42) we now obtain the same quantum equations as above when assuming the “weak gravity” condition (41), which is to say:

$$\begin{aligned}0&=\left(R^{*}{}_{\alpha\beta}-\frac{1}{2}R^{*}\cdot\Omega_{\alpha\beta}\right)\overbrace{\left(\frac{1}{\Psi}\cdot\delta\omega^{\alpha\beta}+\omega^{\alpha\beta}\cdot\delta\left(\frac{1}{\Psi}\right)\right)}^{\delta\Omega^{\alpha\beta}}\\ &=\left(\frac{1}{2\Psi}\left(2\Delta\Psi-2\Psi_{,d}\omega^{cd}{}_{,c}-\frac{n}{(n-1)}\Psi_{,d}\omega^{cd}\omega^{ab}\omega_{ac,b}\right)+\frac{\omega^{ab}\Psi_{,a}\cdot\Psi_{,b}}{4\Psi^{2}}(n-6)-\frac{R}{(n-1)}\right)\end{aligned}, \tag{61}$$

and when further setting:

$$\Psi[\psi]=\psi^{\frac{4}{n-2}}, \tag{62}$$

together with the separation approach from above, we can linearize (61), ending up in the classical Schrödinger hydrogen equation (46). It should be noted that with n being quite huge, the function Ψ would be always close to one, but this does not change the fact that (62) makes (61) to:

$$\begin{aligned}0&=\left(\frac{1}{2\Psi}\left(2\Delta\psi-2\psi_{,d}\omega^{cd}{}_{,c}-\frac{n}{(n-1)}\psi_{,d}\omega^{cd}\omega^{ab}\omega_{ac,b}\right)-\frac{R}{(n-1)}\right)\\ &=\left(\frac{1}{2\Psi}\left(2\Delta\psi-2\psi_{,d}g^{cd}{}_{,c}-\frac{n}{(n-1)}\psi_{,d}g^{cd}g^{ab}g_{ac,b}\right)-\frac{R}{(n-1)}\right)\end{aligned}, \tag{63}$$

where in the second line we have used the fact that:

$$\psi=\psi\left[t,r,\vartheta,\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5\right]. \tag{64}$$

Further assuming the flat space as Schrödinger did and consequently setting R=0 plus applying the separation approach:

$$\psi\left[t,r,\vartheta,\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5\right]=\psi_t\left[t\right]\cdot\psi_r\left[r\right]\cdot\psi_\vartheta\left[\vartheta\right]\cdot\prod_{i=1}^{5}\psi_{\varphi_i}\left[\varphi_i\right], \tag{65}$$

with the partial solutions:

$$\begin{gathered}\psi_{\varphi_i}\left[\varphi_i\right]=\psi_{\varphi_i}\left[g_{\varphi_i}\left[\varphi_i\right]\right]\equiv\psi_{\varphi_i}\left[g_{\varphi_i}\right];\\ \psi_{\varphi_i}\left[g_{\varphi_i}\right]=C_{-i}\cdot e^{-i\cdot A_i\cdot g_{\varphi_i}}+C_{+i}\cdot e^{+i\cdot A_i\cdot g_{\varphi_i}}=C_{ci}\cdot\cos\left[A_i\cdot g_{\varphi_i}\right]+C_{si}\cdot\sin\left[A_i\cdot g_{\varphi_i}\right]\\ \text{especially}\quad(\text{classically}):\\ \psi_{\varphi 1}\left[g_{\varphi_1}\right]=C_{-1}\cdot e^{-i\cdot A_1\cdot g_{\varphi_1}}+C_{+1}\cdot e^{+i\cdot A_1\cdot g_{\varphi_1}}=C_{c1}\cdot\cos\left[\overbrace{A_1}^{\hat{=}m}\cdot g_{\varphi_1}\right]+C_{s1}\cdot\sin\left[\overbrace{A_1}^{\hat{=}m}\cdot g_{\varphi_1}\right]\end{gathered}, \tag{66}$$

$$\psi_t\left[t\right]=C_{t1}\cdot\cos\left[c\cdot E\cdot t\right]+C_{t2}\cdot\sin\left[c\cdot E\cdot t\right], \tag{67}$$

$$\psi_\vartheta\left[\vartheta=\varphi_0\right]=C_{P\vartheta}\cdot P_L^{A_1}\left[\cos\left[\vartheta=\varphi_0\right]\right]+C_{Q\vartheta}\cdot Q_L^{A_1}\left[\cos\left[\vartheta=\varphi_0\right]\right], \tag{68}$$

gives us the following differential equation for the r-dependency of ψ[…]:

$$0=\psi_{\vartheta}\left[\vartheta\right]\cdot\psi\left[\varphi\right]\cdot\begin{pmatrix}\frac{L\cdot(L+1)}{r^{2}}+r^{i\cdot s}\cdot A_{2}^{2}+r^{s}\cdot A_{3}^{2}+r^{-i\cdot s}\cdot A_{4}^{2}\\ -r^{-s}\cdot A_{5}^{2}-\frac{2\partial}{r\cdot\partial r}-\frac{\partial^{2}}{\partial r^{2}}+E^{2}\end{pmatrix}\psi_{r}\left[r\right]\cdot \tag{69}$$

we see that for the choice of s=1 and $\psi_{\varphi_i}\left[\varphi_i\right]=\mathrm{const}$ for i=2,3,4, we obtain:

$$0=\psi_{\vartheta}\left[\vartheta\right]\cdot\psi\left[\varphi\right]\cdot\left(\frac{L\cdot(L+1)}{r^{2}}-\frac{A_{5}^{2}}{r}-\frac{2\partial}{r\cdot\partial r}-\frac{\partial^{2}}{\partial r^{2}}+E^{2}\right)\psi_{r}\left[r\right], \tag{70}$$

just being the radial part of the classical Schrödinger equation for the hydrogen atom.

The delicate part is the linearization condition (62), because there, for huge numbers of n, the scaling function Ψ would always almost be 1. This means that the entanglement of observer and object (here the H-atom) either renders the resulting quantum effect miniscule or leaves in a small non-linearity which does not allow for any superposition of solutions (states). In this case the system has "to make a decision" and chose a state. A simple instability to the dimensionality in the linearization condition (62), potentially caused by a permanent switch of dimensions or a – as one might want to call it – "dimensional uncertainty", would completely suffice to end up in such a non-additive environment. The reason for this unstable behavior may just be that the macroscopic reality of the observer system – being so huge and omnipresently entangled and disentangled with many objects - comes with a permanent change also in its total dimensionality. The observer-object-entanglement then forces this dimensional uncertainty also onto the object being measured.

As said at the beginning of this section, the goal was not to finalize this discussion, but only to point out the connection of the dimensional choice with the measurement process and hence, the ominous "collapse of the wave function".

Extremal Dimensionality for Schrödinger

When now, after almost having lost ourselves in the "collapse of the wave function", coming back to the question of dimensionality of systems, we realize that our n-extremal condition, resulting from a variated volume equation potentially from (32), even a simple quantum problem like the Schrödinger hydrogen results in a rather complicated integral, because we have:

$$\begin{aligned}0&=\delta_{n}\left[\int_{V}d^{n}x\left(\sqrt{-G}\cdot\tilde{\Lambda}\right)\right]=\tilde{\Lambda}\cdot\delta_{n}\int_{V}d^{n}x\sqrt{-G}\\&=\tilde{\Lambda}\cdot\delta_{n}\int_{V}d^{n}x\sqrt{-g\cdot F^{n}}=\tilde{\Lambda}\cdot\delta_{n}\int_{V}d^{n}x\sqrt{-g\cdot f^{\frac{4\cdot n}{n-2}}}\end{aligned}. \tag{71}$$

Using (52) again, we would have to set n=8, but in principle only after the variation, meaning rigorously demanding n-extremality, we would have to leave the dimensionality open and work ourselves through the calculations above with a set of suitable metrics of unknown dimensionality but the characteristic structure bringing about the hydrogen atom Schrödinger equation. In principle this could be done with our observer-object metric (60) where, for the reason of simplicity, we may assume a rather primitive observer consisting only of Cartesian components leading us to:

$$0=\tilde{\Lambda}\cdot\delta_{n}\int_{V}d^{n}x\sqrt{-g\cdot\psi^{\frac{4\cdot n}{n-2}}}\xrightarrow{n\to"\infty"}0=\tilde{\Lambda}\cdot\delta_{n}\int_{V}d^{n}x\cdot\psi^{2}\sqrt{-g}\,. \tag{72}$$

Knowing that the minus sign under the square root in front of the determinant of the covariant metric tensor in principle stands for:

$$0=\delta_{n}\left[\int_{V}d^{n}x\left(\sqrt{-G}\cdot\tilde{\Lambda}\right)\right]=\tilde{\Lambda}\cdot\delta_{n}\int_{V}d^{n}x\sqrt{|G|}\,, \tag{73}$$

gives us from (72):

$$0=\tilde{\Lambda}\cdot\delta_{n}\int_{V}d^{n}x\sqrt{\left|g\cdot\psi^{\frac{4\cdot n}{n-2}}\right|}\xrightarrow{n\to"\infty"}0=\tilde{\Lambda}\cdot\delta_{n}\int_{V}d^{n}x\cdot\psi^{*}\cdot\psi\sqrt{|g|}\,. \tag{74}$$

This part:

$$\int_{V}d^{n}x\cdot\psi^{*}\cdot\psi\sqrt{|g|}=\langle\psi\,|\,\psi\rangle\,, \tag{75}$$

however, is just the quantum mechanical scalar product of the wave function ψ with itself, which, on the right hand side of the equation, we presented in Dirac's bra-ket notation.

Our variational result (74) would then demand the dimensional dependency of this scalar product to be an extremum with respect to the number of dimensions the system lives in.

Conclusions

We found a variety of extremal conditions for various symmetries and objects regarding their dimensionality n. This n-extremality seems to have connections to many aspects in physics. We found a dimensionality rule for the bit-wise growth of black holes in accordance with the Bekenstein-Hawking thought experiment, a limiting equation for the evaluation of the Sommerfeld fine structure constant and a system-size dependency involving the quantum mechanical scalar product for observer-observables-systems. We also saw the aspect of dimensionality of systems in connection with the so-called collapse of the wave-function and tried to explain the process by the means of the entanglement of the observer with the object of interest (being observed).

Appendix: About the Dimensional Size of Systems

In classical systems science there is no way to derive the necessary dimension of a system in a truly fundamental and neutral (mathematically based) manner. Thus, systems are often "defined" as it pleases the creator of the simulation or as there are restrictions in ability and calculation power for the "digital twin" of the natural system one intends to model. As this holds for any system, this is also true – of course – for the unconscious or conscious mind and thus, of great interest here.

We start with the conjecture that not just the system's inner properties and corresponding governing equations but also the system's size (number of degrees of freedom or dimensions) can be derived from a suitable minimum principle. Our starting point shall this time be the Einstein-Hilbert action with a generalized Lagrange density function $\Phi_{R}[R]$ and a yet undefined variation, which we write as follows:

$$\delta_{?}W=0=\delta_{?}\int_{V}d^{n}x\left(\sqrt{-g}\cdot\Phi_{R}[R]\right). \tag{76}$$

Please note that we could also write this for a scaled metric tensor as elaborated in the previous appendices in order to work out the connection to quantum theory, respectively, in order to make it show itself directly via a set of wrapping and wave functions $F_i[f_i]$ and f_i within the usual variational calculus.

In order to adjust undefined parameters and finalize the character of the variational task (77), we intend to consider a fundamental problem and here determine the size of a black hole in a completely new way. Classically the size of a black hole is given by the Schwarzschild radius, which itself is determined by the mass m of the black hole via: $r_s = \frac{2 \cdot m \cdot G}{c^2}$ (G… Newton's constant, c… speed of light in vacuum). This Schwarzschild radius, however, was never derived from a first principle, but was adjusted as a parameter to the Schwarzschild metric [14] in order to give the correct limit to the Newton gravitational law.

Here now we want to derive the Schwarzschild radius via a suitable version of (77). In order to do so, we first need to repeat Bekenstein's thought experiment of black holes.

The Bekenstein Bit-Problem

One of the most famous and equally puzzling problems in General Theory of Relativity is the Bekenstein-Bit problem, where it was found that black holes can store information, but so far it is been seen as a mystery how these objects actually do this. In [2, 7, 8] we have shown that bit-like information is been stored as dimensions and that each bit becomes one dimension. For convenience we are here repeating parts of the original evaluation.

In the early seventies J. Bekenstein [3, 4] investigated the connection between black hole surface area and information. Thereby he simply considered the surfaces change of a black hole which would be hit by a photon just of the same size as the black hole. His idea was that with such a geometric constellation the outcome of the experiment would just consist of the information whether the photon fell into the black hole or whether it did not. Thus, it would be a 1-bit information. His calculations led him to the funny proportionality of area and information. He found that the number of bits, coded by a certain black hole, is proportional to the surface area of this very black hole if measured in Planck area ℓ_P^2. In fact, the dependency how one bit of information changes the area of the black hole (ΔA) reads:

$$\Delta A = 32 \cdot \pi^2 \cdot \ell_P^2 + 64 \cdot \pi^3 \cdot \frac{\ell_P^4}{r_s^2}. \tag{77}$$

Thereby the derivation of this equation is performed as follows. At first we start with the assumption that the photon's right size should be a wavelength λ of the Schwarzschild radius r_s. Knowing that the energy of the photon would be E=h*ν, with denoting ν the frequency and h giving the Planck constant, and plugging in the equation for Schwarzschild radius of the photon related mass change Δm (with reduced Planck constant $\hbar$ and the Newton constant G):

$$\begin{aligned} &\frac{\Delta r_s \cdot c^4}{2G} = \Delta m \cdot c^2 = \leftarrow \boxed{E = h \cdot \nu} \rightarrow = \frac{h \cdot c}{\lambda} = \frac{h \cdot c}{r_s} \\ &\Rightarrow \frac{\Delta r_s \cdot c^4}{2G} = \frac{h \cdot c}{r_s} \Rightarrow \Delta r_s \cdot r_s = 2\frac{h \cdot G}{c^3} = 4\pi \frac{\hbar \cdot G}{c^3} = 4\pi \cdot \ell_P^2 \Rightarrow \Delta r_s = 4\pi \cdot \frac{\ell_P^2}{r_s}, \end{aligned} \tag{78}$$

we can derive ΔA as follows:

$$\Delta A = 4\pi\left(\left(\Delta r_s + r_s\right)^2 - r_s^2\right) = 4\pi\left(2\Delta r_s \cdot r_s + \left(\Delta r_s\right)^2\right)$$
$$= 32 \cdot \pi^2 \cdot \ell_P^2 + 64 \cdot \pi^3 \cdot \frac{\ell_P^4}{r_s^2} \quad . \tag{79}$$

Ignoring the extremely small second term in the last line, one could just assume our black hole to be constructed of many such bit surface pieces. Thus, we could write:

$$q \cdot \Delta A = q \cdot 32 \cdot \pi^2 \cdot \ell_P^2 = 4 \cdot \pi \cdot r_s^2 \quad \Rightarrow \quad r_s^2 = q \cdot 8 \cdot \pi \cdot \ell_P^2, \tag{80}$$

where r_s gives the radius of the black hole. We see that our black hole radius is proportional to the square root of the bits q thrown into it.

Now we want to compare the dependency $r_s[q]$ with the radii $r_{max}[N]$ resulting in maximum volume of n-spheres for a certain number of space-time dimensions N=n+1.

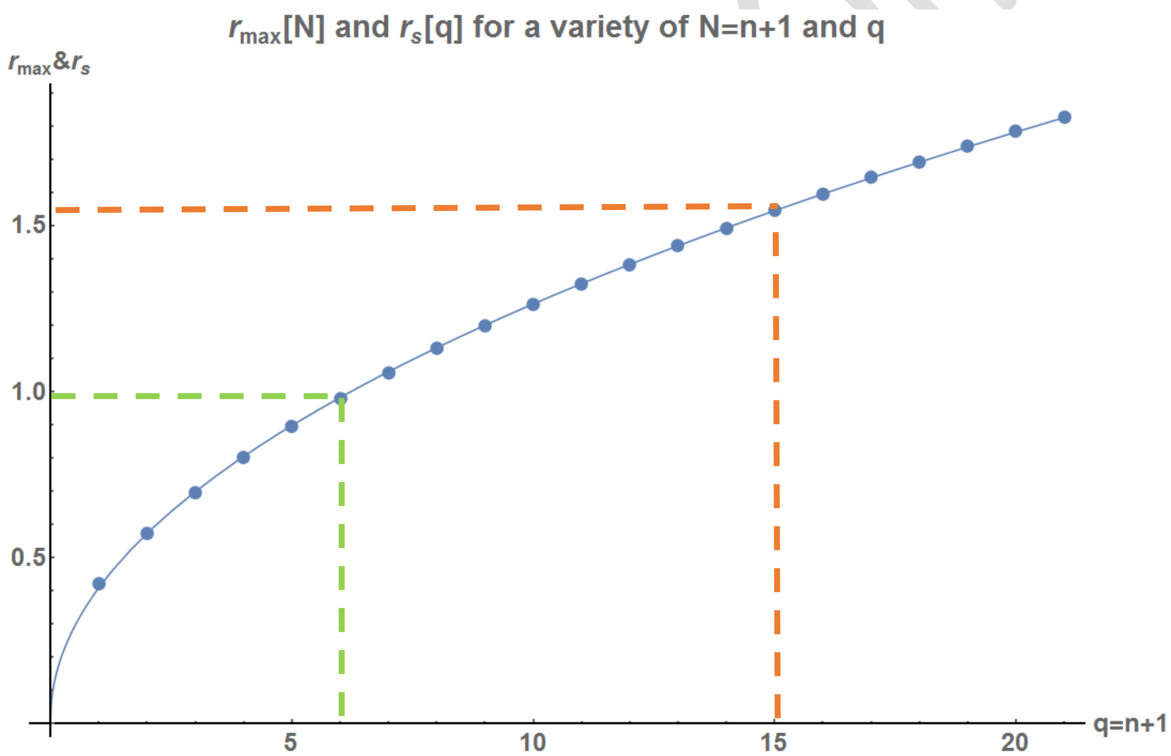


Fig. A1: Radius r_{max} for which at a certain number of dimensions the n-sphere has maximum volume in dependency on N=n+1 compared with the increase of the Schwarzschild radius r_s of a black hole in dependence on the number of bits q thrown into it. We find that q=N=n+1. As examples we pick the situation with a radius slightly bigger than 1.5 (whatever unit). We obtain maximum volume for a sphere in 15 dimensions (orange dotted line). Picking a radius slightly below 1, however, gives us a 6-dimensional sphere which can have maximum volume at such a size (green dotted line).

We find a perfect fit (s. figure A1) to the $r_{max}[N]$-dependency for q=N with the following function:

$$r_s = U \cdot \left(0.014948 + 0.3951244 \cdot \sqrt{q}\right); \quad U^2 = 8 \cdot \pi \cdot \ell_P^2, \tag{81}$$

where U denotes a unit-factor which was set U=1 in figure A1.

Our finding does not only connect the intrinsic dimension of a black hole with its mass respectively its surface, but also, at least partially, gives an explanation to the hitherto unsolved problem of "what are the micro states of a black hole giving it temperature and allowing it to store information". According to the evaluation in this section, these microstates are just various states of dimensions realized within the black hole in dependence on the number of bits it contains (and thus, its mass). The bigger the number of bits, the higher the intrinsic dimensions the black hole has. In fact, the connection even is a direct one and only seems to deviate from the simple direct proportionality for very low numbers of masses[2], respectively Schwarzschild radii r_s, respectively numbers of bits q the black hole has swallowed.

This finding also gives us a direct connection between a principle mathematical law (the maximum volume as function of the dimension for a given radius of an n-sphere) to the number of bits a black hole contains, to the mass or Schwarzschild radius of this very black hole and the number and character of microstates the black hole actually uses to internally code the bits.

It has to be pointed out that the expression "intrinsic dimension" truly stands for the part of space for $r<r_s$, which is to say, the space behind the event horizon. As for the outside, the solution of a Schwarzschild object in n+1-dimensional space-times is given via:

$$g_{\alpha\beta}^{N}=\begin{pmatrix} -c^2\cdot f[r] & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{f[r]} & 0 & 0 & \cdots & 0 \\ 0 & 0 & r^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & r^2\cdot\sin^2\varphi_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{aa} \end{pmatrix}$$

$$g_{44}=r^2\cdot\sin^2\varphi_1\cdot\sin^2\varphi_2;\quad g_{aa}=r^2\cdot\prod_{j=1}^{a-2}\sin^2\varphi_j;\quad a=N-1=n;\cdot \tag{82}$$

$$f[r]=1-\frac{r_s^{N-3}}{r^{N-3}}=1-\frac{r_s^{n-2}}{r^{n-2}}$$

As we find that the Newton laws of gravity, however, require N=4, it has to be assumed that in a region near the event horizon the dimension of the black hole decreases to the known 4 dimensions so that Newton's laws of gravity are properly mirrored to us (as outside observers). The corresponding derivation was given in [8] (chapter 11).

A System-Immanent Scale

Note: the correct solution for the evaluation of the Schwarzschild radius r_s as function of the bits thrown into a black hole object (if using the results from [15]) would be:

$$r_s=2\cdot\ell_P\cdot\sqrt{\pi\cdot\left(q+\sqrt{q\cdot(1+q)}\right)}\,! \tag{83}$$

We find a perfect fit to the n-spheres with maximized volume to a given radius (dots in figure A1) with a Planck length of $\ell_P=0.07881256452824544$ (s. figure A2, which is almost perfectly equal to the fit in figure A1).

[2] Besides, this deviation is also suggested by the Bekenstein finding summed up in equation (54), where we could assume the second term to become of importance at lower numbers of r_s.

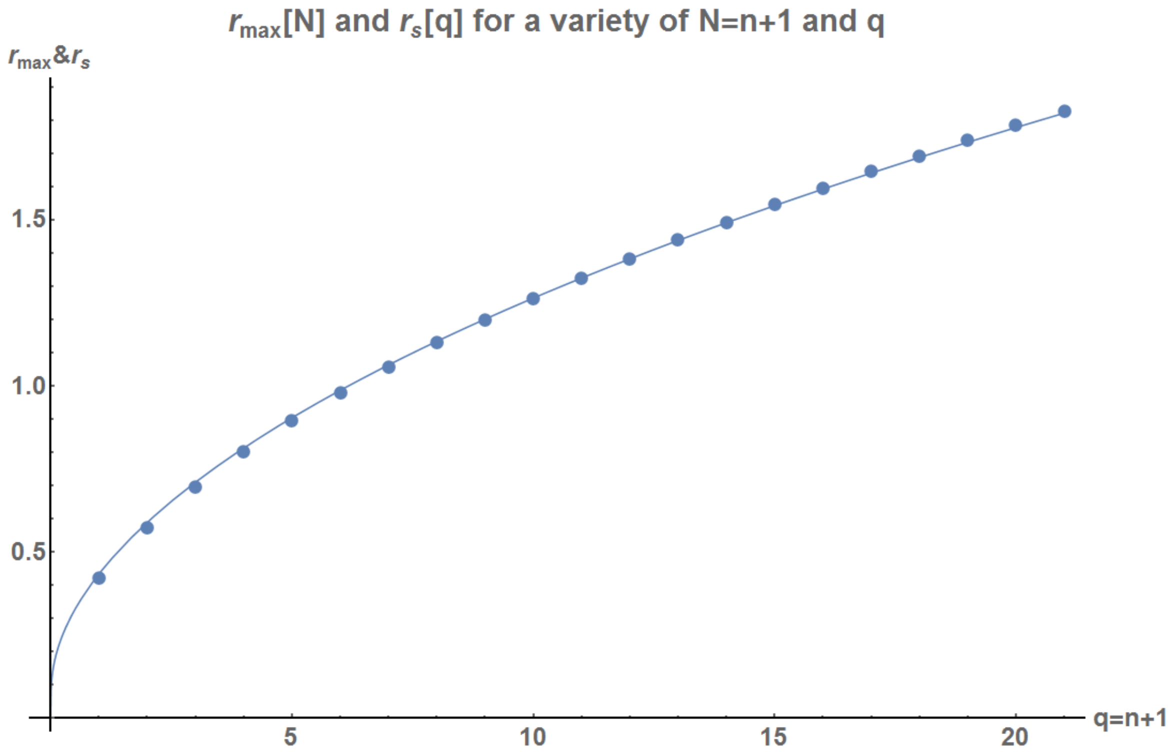


Fig. A2 (Please note that we have applied a slightly different fit than it was applied in fig. A1): Radius r_{max} for which at a certain number of dimensions the n-sphere has maximum volume in dependence on N=n+1 compared with the increase of the Schwarzschild radius r_s of a black hole in dependence on the number of bits q thrown into it by using (84). We find that q=N=n+1.

But what would be the unit for this Planck length?

Well, it was already shown in [8] that by using the results from [15] and the "volume integral" (77) with:

$$\delta_n W = 0 = \delta_n \int_V d^n x \sqrt{-g} \,, \tag{84}$$

for n-spheres (in [15] with T[n]=1):

$$V_N = V_{n+1} = T[n] \cdot \frac{\pi^{\frac{(n)}{2}}}{\Gamma\left[\frac{(n+2)}{2}\right]} \cdot r^n \,, \tag{85}$$

we have evaluated the very dimension n to which at a given radius r_s the volume of a n-sphere has a maximum. The r_s are put into the calculation as plain "natural" numbers, meaning an r_s=1 is just a 1 and that is it. We might name this unit a "mathematical meter" or just "mams" (plural for "mathematical meters"). Transformation to our usual units, like meters, requires the introduction of a factor $T[n]=U^n$.

With $U = \frac{[\ell_P]_{in\_meters}}{[\ell_P]_{in\_mams}} = \frac{1.616255(18)\times 10^{-35}\text{ meters}}{0.07881256452824544}$, for instance, we can easily change to our meters. Nevertheless it appears somehow astonishing that there seems to exist a fundamental "natural" unit, being completely based on a mathematical - geometrical - extremal principle (the maximum volume of n-spheres as functions of their dimensions for certain radii). It is also interesting that these dimensions are so nicely correlated to the number of bits a black hole has swallowed. In fact, using the unit of mams, the number of spatial (n-sphere) dimensions is perfectly equal to the number of swallowed bits.

Thus, in the case that black holes would in fact store their content as dimensions and the Einstein-Hilbert-Action being extended with respect to the number of dimensions in addition to the metric, we immediately also get an absolute scale for our black hole system in which the number 1 is “made out” of 12.6883 Planck length and where a 3-sphere has a radius of 0.6969979737167096 mams.

Back to the Optimum Size Question for any System

When observing the integral in (77), we see that – in principle – we seek for a maximum volume for a given dimension or, taking the radius of a Schwarzschild object, look for the corresponding dimension making the volume integral an extremum. As the determinant g of the Schwarzschild metric is just equal to the one of a n-sphere with the additional time-dimension to be integrated, we can easily use the volume integral result of n-spheres, which reads:

$$V_N = V_{n+1} = \frac{\pi^{\frac{(n)}{2}}}{\Gamma\left[\frac{(n+2)}{2}\right]} \cdot r^n . \tag{86}$$

Please note that due to the time-coordinate t we have V_{n+1} instead of V_n. Thereby the integration via t in (87) is assumed to be performed such that it would give 1. In general, we might take care about this part of the integration via a proportional constant T we could even consider to be n-dependent T[n] and thus, equation (86):

$$V_N = V_{n+1} = \overbrace{T[n]}^{=U^n} \cdot \frac{\pi^{\frac{(n)}{2}}}{\Gamma\left[\frac{(n+2)}{2}\right]} \cdot r^n . \tag{87}$$

Now we evaluate the various dimensions for which, for a given radius r of the n-sphere, we would obtain extrema. The results were already given in figures A1 and A2. There we have illustrated the resulting r_{max} as functions of the dimensions N=n+1 (note: n=n-sphere dimension, N=t+n-sphere dimension).

Now we just compare our findings with the original question of extracting a minimum principle for the dimensional size of a given system with our generalized starting point for the variational task (77) and conclude that:

$$\begin{gathered} \delta_? W = 0 = \delta_? \int_V d^n x \left(\sqrt{-g} \cdot \Phi_R[R]\right) \\ \Rightarrow \delta_n W = 0 = \delta_n \int_V d^n x \left(\sqrt{-g} \cdot \left[\Phi_R[R] = 1\right]\right) = \delta_n \int_V d^n x \sqrt{-g} \end{gathered} . \tag{88}$$

Thus, the determination of the optimum size of a system we intend to consider, investigate or analyze can just be found by a dimensional variation of the volume integral of that very system. In the case of spherical symmetries, this then leads to equations of the form:

$$\Rightarrow \delta_n W = 0 = \delta_n \int_V d^n x \left(\sqrt{-g} \cdot \left[\Phi_R [R] = 1 \right] \right) = \delta_n T[n] \cdot \frac{\pi^{\frac{(n)}{2}}}{\Gamma\left[\frac{(n+2)}{2}\right]} \cdot r^n . \tag{89}$$

Along the way we also can extract suitable fundamental scales for our system.

The First Bit Requires the Highest Mass = The First Thought is the Most Difficult

From (84) we can now extract the minimum Schwarzschild radius for the storage of one bit, which would be equal to 5.508 Planck length and corresponds to 11.16 times the Planck mass. This is a huge amount of mass and thus, also energy, one needs to safely store just one single bit. Luckily, the situation improves the more bits one intends to store, as for instance the one millionth bit only requires about $5.5 \cdot 10^{-3}$ Planck masses. Please note that, of course, one might also store bits within spin arrangements of electrons. Then a 1-bit information would be connected with a single electron, whose mass and spin energy is many magnitudes below the Planck mass. This spin storage, however, cannot be seen as a storage of a classical binary bit, because in fact it resembles a quantum bit. Apparently, the safe storage of a pure and truly binary information requires an - almost - macroscopic massive structure. Here the black hole probably provides the smallest possible mass ensemble there can be to arrange such a storage for a certain bit. The limit is given at about 11 times the Planck mass. Only from this mass onward black holes can store binary information… at least until the Hawking radiation leads to a destruction of our black hole 1-bit storage system.

And what would then happen to the stored information? Well, this brings us to the question how safe is information within our universe [16].

Byproducts: A few Fundamental Questions

About the Relativity of System-Scales

We saw that – similar to the Bekenstein or Bekenstein-Hawking problem (see reference [17]) – we add bits via dimensions to our metric structure (here a black hole). Assuming our metric system to be a black hole (which we here only use to have a simple as possible math), we can even obtain a ratio of the black hole's radius r_s to the smallest structure this black hole can resolve. Taking the result for ℓ_P - the Planck length - from the Bekenstein thought experiment, we find the ratio between the Schwarzschild radius r_s of a black hole and ℓ_P:

$$\frac{r_s}{\ell_P} = 2 \cdot \sqrt{\pi \cdot \left(q + \sqrt{q \cdot (1+q)} \right)} . \tag{90}$$

In our universe the Planck length ℓ_P is considered to be the smallest length possible to resolve. What if the ratio (91), we found for black holes, actually is a more fundamental law? At any rate, it appears logic to assume that more bits could be coded or stored by an object of bigger size and smaller internal structure, thereby leaving more options to describe something with these structures. Thus, the equation (91) makes intuitive sense, but could it also be just the other way round? Could it be that to an object of given size the number of bits (being equivalent to its dimensions as we see in figures A1, A2) it contains, determines the smallest scale – the Planck length – of the object, too? And, if referring to the "Mathematical Psychology", the system presents a thinking entity, does this also mean that thoughts have a physical scale?

From inside and taking the Planck length as measure, the increase of information to this very object, subject or entity would look like an increase of its size. Now assuming the inside of the black hole to be a general system, the inhabitants of this system may see this system as their very own universe and would register the increase of information as a growth of their “universe”, measured in the Planck length of that very system-universe. When learning, we seem to feel the increase of mind. May be this perception is just what is actually really going on.

Does More Information Always Mean More Mass?

Quantum computer scientists have already pointed out that, with our current way of storing information, we will one day reach a limit with respect to the number of atoms we can apply for the storing process and the energy being needed to keep the information stable (maintained). Citing from the abstract of [18], we have the following situation:

“Currently, we produce ~10^{21} digital bits of information annually on Earth. Assuming a 20% annual growth rate, we estimate that after ~350 years from now, the number of bits produced will exceed the number of all atoms on Earth, ~10^{50}. After ~300 years, the power required to sustain this digital production will exceed 18.5 × 10^{15} W, i.e., the total planetary power consumption today, and after ~500 years from now, the digital content will account for more than half Earth’s mass, according to the mass-energy–information equivalence principle. Besides the existing global challenges such as climate, environment, population, food, health, energy, and security, our estimates point to another singular event for our planet, called information catastrophe.”

It has to be pointed out that when looking for possible inner Schwarzschild solutions [19], we also found that there are solutions, where the mass decreases with the increase of the object size. It may well be that such strange states are not only realized in black holes, but could perhaps also help to overcome our future information storage problem.

Generalization to General Spheres?

In the sub-sections above we saw that, when taking the equation for the Laplace length ℓ_P from the Bekenstein thought experiment [3, 4]), we find the following ratio between r_s (Schwarzschild radius of a black hole) and ℓ_P (c.f. equation (91)):

$$\frac{r_s}{\ell_P} = 2\cdot\sqrt{\pi\cdot\left(q+\sqrt{q\cdot(1+q)}\right)}\,. \qquad (91)$$

Most interestingly, we also found that the solution to the extremal volume problem for a fixed radius r_f for n-spheres results in the same dependency when variating with respect to the number of dimensions of those n-spheres. We obtain (see dots in figures A1 and A2) excellent fits, when applying an approach like:

$$r_f = L\cdot\sqrt{\pi\cdot\left(n+\sqrt{n\cdot(1+n)}\right)}\,. \qquad (92)$$

Thereby we have the characteristic (system-dependent) length scale L.

This automatically gives us a connection between the size-parameter r_f of any system of spherical symmetry and its theoretical capability to store information. A perfect mathematical n-sphere thereby follows the rule (93) almost perfectly, while other systems may do so only from certain critical sizes onwards, but, nevertheless, we think we can draw the conclusion that the information storage capacity of given systems – if showing enough spherical symmetry – can be extracted from (93). Then the structural size-parameter r_f determines the number of storable bits n in dependence on the system-immanent length parameter L.

From this, one even may deduce that r_f and L could be substituted by other system characteristics. While in (93) their dimension is length, we should not exclude mass, time, charges, energies and so on.

Consequences from the Bekenstein Thought Experiment Regarding the Solutions to the Quantum-Einstein-Field-Equations

In [2, 8] we have shown that the classical n-dimensional Schwarzschild solution could be applied to construct internally structured n-dimensionally black holes, while outside we still have the usual 4-dimensional solution from [14] with the classical Schwarzschild metric. This, however, would not explain how the black hole can code any information.

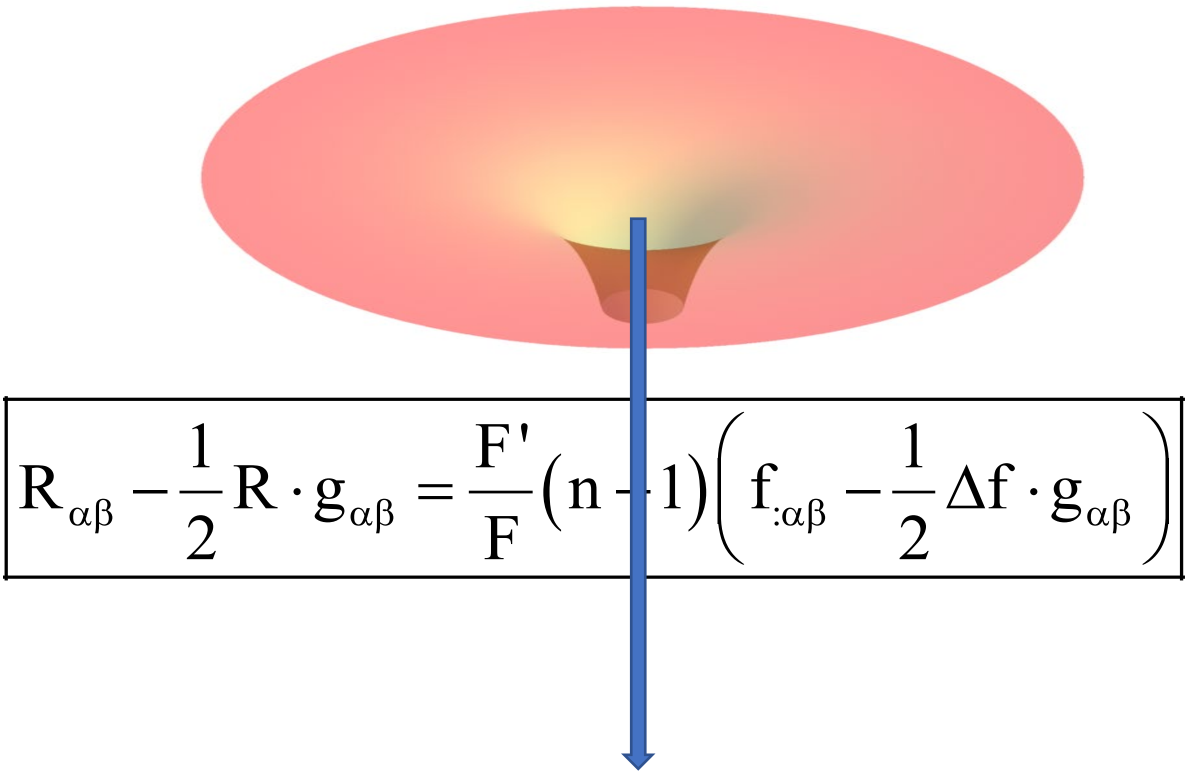
With the help of the new metric solutions evaluated in [19], namely, to just give an example, in the three-dimensional case with coordinates t, r and an angle:

$$g_{\alpha\beta} = C_1 \cdot f[t]^4 \cdot \begin{pmatrix} -c^2 & 0 & 0 \\ 0 & t^2 & 0 \\ 0 & 0 & t^2 \cdot \sin(r)^2 \end{pmatrix}, \qquad (93)$$
$$f[t] = \sqrt{t^{\pm i \cdot c - 1}} \cdot C_f$$

(please note that r has become an angle while t took over the position of the radius),

$$g_{\alpha\beta} = C_1 \cdot f[t]^4 \cdot \begin{pmatrix} -c^2 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \cdot \sin(r)^2 \end{pmatrix} \qquad (94)$$
$$f[t] = e^{\pm \frac{i \cdot c}{2 \cdot \rho} \cdot t} \cdot C_f$$

(this represents a shell-like object) we want to solve also this problem. A generalization of this type solution is been given in appendix N of [2]. Thereby, we found that the Schwarzschild singularity could be avoided (fig. A3).



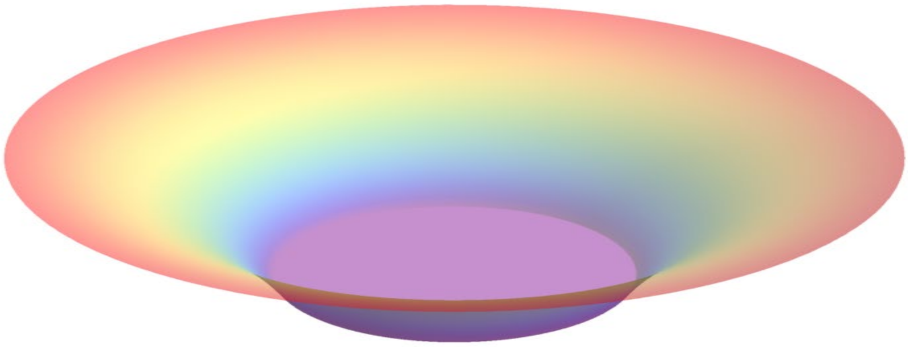


Fig. A3: "From the classical Schwarzschild solution to a Quantum Black Hole" [19]

At first, however, we should note that also the n-dimensional Schwarzschild solutions from [8], section 3.8 (c.f. solution (83) in here) would provide plenty of options to code information, because there are enough degrees of freedom regarding the thicknesses of the individual x-dimensional layers of the onion-like Schwarzschild object, which was proposed there (see also [19]). Similar assumptions could be made for the Robertson-Walker approach introduced in [19], but apart from again mentioning the onion-layer structured ogre-mind from the movie Schreck, we will not further consider these possibilities in here.

In the case of photonic inner solutions as also suggested in [19] one might assume some kind of standing waves inside the black hole, but as we currently don't have the math to realize such structures, we postpone the investigation of this possibility.

Thus, we here concentrate on solutions (94), (95) as potential inner solutions to a black hole. As we see that the parameter ρ clearly is a length, we want to derive its properties. For the general case this was already done in appendix N. Nevertheless, we repeat it here for the setting (94) and (95). From basic quantum theory we know that a particle at rest has the time dependency:

$$f[t]=e^{\pm i\cdot\frac{m\cdot c^2}{\hbar}\cdot t}\cdot C_f\,, \tag{95}$$

with m giving the rest mass of the particle and $\hbar$ denoting the reduced Planck constant. Comparing with the f[t]-function from the metric solution (95), we find:

$$\frac{m\cdot c}{\hbar}=\frac{1}{2\cdot\rho}. \tag{96}$$

Inserting the Schwarzschild radius $r_s=\frac{2\cdot m\cdot G}{c^2}$ (G… Newton's constant, c… speed of light in vacuum), thereby substituting the rest mass m, leaves us with:

$$\frac{r_s\cdot c^3}{2\cdot\hbar\cdot G}=\frac{1}{2\cdot\rho}\quad\Rightarrow\quad\rho=\frac{1}{r_s}\cdot\left(\frac{c^3}{\hbar\cdot G}\right)^{-1}=\frac{\ell_P^2}{r_s}. \tag{97}$$

Here ℓ_P denotes the Planck length. By inserting (91) into (98) we obtain:

$$\rho=\frac{\ell_P^2}{r_s}=\frac{\ell_P^2}{2\cdot\ell_P\cdot\sqrt{\pi\cdot\left(q+\sqrt{q\cdot(1+q)}\right)}}=\frac{\ell_P}{2\cdot\sqrt{\pi\cdot\left(q+\sqrt{q\cdot(1+q)}\right)}}. \tag{98}$$

Thus, while for a black hole the number of bits thrown into it leads to an almost perfectly square-root-like increase of the Schwarzschild radius in accordance with equation (91), the ρ-parameter of the (95)-objects decreases with the number of bits. (95)-objects would have the same ρ-parameter, which we may see as a size, as a black hole only for Schwarzschild radii r_s equal to the Planck length. In other words, for growing black holes with radii bigger than the Planck length the corresponding equally heavy (95)-objects would be significantly smaller than the black holes.

So, we ask: Could the (95)-objects be used as building blocks for the black holes, residing inside it, which is to say behind the event horizon?

Assuming that the black hole's surface is made out of metric spherical objects of the type (95) and further assuming that each of these objects in the surface of the black hole, which is to say at $r=r_s$ (which also happens to be the event horizon), requires its own surface space of something like $C_\rho * \rho^2$, we can directly evaluate the number of such (95)-objects, we from now on name ρ-spheres, are residing inside the event horizon with increasing numbers of bits thrown into the black hole. Assuming that the mass is always additive, the total mass m of the black hole must then be distributed among the N ρ-spheres, which changes (98) to:

$$\frac{r_s \cdot c^3}{2 \cdot \hbar \cdot G} = \frac{1}{2 \cdot N \cdot \rho} \quad \Rightarrow \quad N \cdot \rho = \frac{1}{r_s} \cdot \left(\frac{c^3}{\hbar \cdot G}\right)^{-1} = \frac{\ell_P^2}{r_s} \quad \Rightarrow \quad \rho = \frac{\ell_P^2}{N \cdot r_s}. \tag{99}$$

Also having to satisfy the following equation for the N ρ-spheres sitting on the surface, we have to solve the following equation:

$$\begin{aligned} N \cdot C_\rho \cdot \rho^2 &= 4 \cdot \pi \cdot r_s^2 \\ \Rightarrow C_\rho \cdot \frac{\ell_P^2}{4 \cdot \pi \cdot N \cdot \left(q + \sqrt{q \cdot (1+q)}\right)} &= (4 \cdot \pi)^2 \cdot \ell_P^2 \cdot \left(q + \sqrt{q \cdot (1+q)}\right). \\ \Rightarrow N &= \frac{1}{64 \cdot \pi^3 \cdot \left(q + \sqrt{q(1+q)}\right)^2} \end{aligned} \tag{100}$$

We realize, that such a structure could not be used to store any information, because the number of ρ-spheres should have to increase with the number of bits and not decrease as it does. Things are improving the moment we allow a combination of ρ-spheres and (94)-objects (the latter we shall call t-spheres) to make up our inner black hole. We propose the following (simplest of the many possibilities) structure:

A) In the center of the black hole sits a ρ-spheres of "radius-parameter" ρ given in (98) and thus, $\rho = \frac{\ell_P^2}{r_s}$, which is to say, the bigger the Schwarzschild radius r_s of the black hole, the smaller its core. In fact, for infinite masses the core would become a singularity.
B) This single ρ-sphere core is surrounded by t-spheres (94) and the number of those t-spheres, which a black hole can bind, is proportional to the number of bits the black hole has swallowed.
C) Taking the Bekenstein-condition, this demands an average size for the t-spheres, being bound by the black hole or the black hole's surface, to be such that its projected surface would be equal to ℓ_P^2. In other words, we could assume the average radius of the t-spheres (the ones bound to the black hole) to be equal to $\ell_P / \sqrt{\pi}$.

With such a structure, it is very well possible that in fact black holes have no singularity and follow our scheme of inner-outer-solution, but one cannot detect any difference to the classical

Schwarzschild solution from the outside, because the inner-parts are always hidden behind the event horizon.

But does this help us to solve the Bekenstein information problem?

Yes, it does.

We can imagine many t-sphere objects (of number N=q) sitting on the surface of the black hole. As the generalized solution to (94) would read:

$$g_{\alpha\beta} = C_1 \cdot f[t]^4 \cdot \begin{pmatrix} -c^2 & 0 & 0 \\ 0 & A^2 \cdot t^2 & 0 \\ 0 & 0 & B^2 \cdot t^2 \cdot \sin(r)^2 \end{pmatrix}, \qquad (101)$$
$$f[t] = \sqrt{t^{\pm\frac{i\cdot c}{A}-1}} \cdot C_f$$

we see that each t-sphere could not only store information via a certain sign within the exponent, but also via the free parameter B.

The Size of the Electron?

Applying (100) and assuming a ρ-sphere-structure for the electron, gives us:

$$\frac{r_s \cdot c^3}{2 \cdot \hbar \cdot G} = \frac{1}{2 \cdot N \cdot \rho} \quad \Rightarrow \quad N \cdot \rho = \frac{1}{r_s} \cdot \left(\frac{c^3}{\hbar \cdot G} \right)^{-1} = \frac{\ell_P^2}{r_s}$$
$$\Rightarrow \quad \rho = \frac{\ell_P^2}{N \cdot r_s} = \frac{1.93 \times 10^{-13}}{N} \text{meter} \qquad (102)$$

Setting N=1 we would end up with a ρ-sphere of $\rho = 1.93 \times 10^{-13}$ meter for the "pure" or "naked" electron.

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