The Special Case of Spatial Dimensionality: n=3

By Dr. rer. nat. habil. Norbert Schwarzer

Abstract

In this paper we are going to derive a quantum gravity equation from which the preference of 3 spatial dimensions falls out.

Introduction

After we saw that the number of dimensions n for certain systems also seems to be determined by an extremal principle [1], we want to find out under what conditions we would obtain a rule for n=3.

Quantum Gravity or the "Theory of Everything"

We start with the following scaled metric tensor and force it into the Einstein-Hilbert action [2] variational problem as follows:

$$G_{\alpha\beta} = g_{\alpha\beta} \cdot F[f] \quad \rightarrow \quad \delta W = 0 = \delta \int_V d^n x \sqrt{-G} \cdot R^* \tag{1}$$

Here G denotes the determinant of the metric tensor from (1) and R^* gives the corresponding Ricci scalar. Performing the variation with respect to the metric $G_{\alpha\beta}$ results in:

$$0 = \left(\begin{aligned} & \boxed{R_{\alpha\beta} - R\frac{g_{\alpha\beta}}{2}} \\ & -\frac{1}{2F}\left(\begin{aligned} & F_{,\alpha\beta}(n-2) + F_{,ab} g_{\alpha\beta} g^{ab} + F_{,a} g^{ab}\left(g_{\beta b,\alpha} - g_{\beta\alpha,b}\right) - \\ & F_{,\alpha} g^{ab} g_{\beta b,a} - F_{,\beta} g^{ab} g_{\alpha b,a} + F_{,d} g^{cd} \left(\begin{aligned} & g_{\alpha c,\beta} - \frac{1}{2} n g_{\alpha c,\beta} - \frac{1}{2} n g_{\beta c,\alpha} \\ & + \frac{1}{2} n g_{\alpha\beta,c} + \frac{1}{2} g_{\alpha\beta} g_{ab,c} g^{ab} \end{aligned} \right) \end{aligned} \right) \\ & + \frac{1}{4F^2}\left(F_{,\alpha} \cdot F_{,\beta}(3n-6) + g_{\alpha\beta} F_{,c} F_{,d} g^{cd} (4-n) \right) \\ & + (n-1)\left(\frac{1}{2F}\left(\begin{aligned} & 2\Delta F - 2F_{,d} g^{cd}{}_{,c} \\ & - \frac{n}{(n-1)} F_{,d} g^{cd} g^{ab} g_{ac,b} \end{aligned} \right) + \frac{g^{ab} F_{,a} \cdot F_{,b}}{4F^2}(n-6) \right) \cdot \frac{g_{\alpha\beta}}{2} \end{aligned} \right) \delta G^{\alpha\beta} \tag{2}$$

and shows us that we have not only obtained the classical Einstein Theory of General Relativity [3] (see boxed terms exactly giving the Einstein Field Equations in vacuum), but also a set of quantum field equations for the scaling function F, clearly playing the role of the wave-function. It was shown in our previous publications [4, 5, 6, 7, 8] that these additional terms are quantum equations, fully covering the main aspects of relativistic classical quantum theory. So, we conclude, that we indeed have a Quantum Gravity Theory or Theory of Everything, as one also calls it, at hand, whereby it should be pointed out that (2) has to be considered the simplest possible – and still general (see [4, 5, 6, 7, 8]) - form for the corresponding quantum gravity field equations.

Conditions for n=3

The “Führungsfeld” or wave function F[f] shall be fixed such that the scalar non-linear terms (times metric tensor) in the whole Einstein tensor (c.f. (2) and (3) with the red terms representing or influencing this condition) vanishes:

$$0=\left(\begin{array}{c}\boxed{\boxed{R_{\alpha\beta}-R\frac{g_{\alpha\beta}}{2}}}\\ -\frac{1}{2F}\left(\begin{array}{c}F_{,\alpha\beta}(n-2)+F_{,ab}g_{\alpha\beta}g^{ab}+F_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-\\ F_{,\alpha}g^{ab}g_{\beta b,a}-F_{,\beta}g^{ab}g_{\alpha b,a}+F_{,d}g^{cd}\left(\begin{array}{c}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)\end{array}\right)\\ +\frac{1}{4F^{2}}\left(F_{,\alpha}\cdot F_{,\beta}(3n-6)+g_{\alpha\beta}F_{,c}F_{,d}g^{cd}(4-n)\right)\\ +(n-1)\left(\frac{1}{2F}\left(\begin{array}{c}2\Delta F-2F_{,d}g^{cd}{}_{,c}\\ -\frac{n}{(n-1)}F_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right)+\frac{g^{ab}F_{,a}\cdot F_{,b}}{4F^{2}}(n-6)\right)\cdot\frac{g_{\alpha\beta}}{2}\end{array}\right)\delta G^{\alpha\beta} \tag{3}$$

Consequently, our setting for F(f) shall be:

$$F[f]=\begin{cases}C_F\cdot\left(f+C_f\right)^{\frac{4}{n-3}} & n\neq 3\\ C_F\cdot e^{f\cdot C_f} & n=3\end{cases}. \tag{4}$$

In this case (2) yields the following field equations:

$$0=\left(R^*_{\alpha\beta}-R^*\frac{1}{2}G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}$$

$$=\left(\begin{array}{c}\left(R_{\alpha\beta}-\frac{F'}{2F}\begin{pmatrix}f_{,\alpha\beta}(n-2)+f_{,ab}g_{\alpha\beta}g^{ab}+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-f_{,\alpha}g^{ab}g_{\beta b,a}\\ -f_{,\beta}g^{ab}g_{\alpha b,a}+f_{,d}g^{cd}\begin{pmatrix}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{pmatrix}\end{pmatrix}\right)\\ -\left(R-\frac{F'}{2F}\begin{pmatrix}(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)\\ -nf_{,d}g^{cd}g^{ab}g_{ac,b}\end{pmatrix}\right)\cdot\frac{1}{2}g_{\alpha\beta}\\ +\frac{1}{4F^2}f_{,\alpha}\cdot f_{,\beta}(n-2)\left(3(F')^2-2FF''\right)\end{array}\right)\delta G^{\alpha\beta}$$

$$=\left(\begin{array}{c}\left(\begin{array}{c}R_{\alpha\beta}-\frac{2}{(n-3)(f+C_f)}\\ \times\begin{pmatrix}f_{,\alpha\beta}(n-2)+f_{,ab}g_{\alpha\beta}g^{ab}+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-f_{,\alpha}g^{ab}g_{\beta b,a}\\ -f_{,\beta}g^{ab}g_{\alpha b,a}+f_{,d}g^{cd}\begin{pmatrix}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{pmatrix}\end{pmatrix}\end{array}\right)\\ -\left(R-\frac{2}{(n-3)(f+C_f)}\times\begin{pmatrix}(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)\\ -nf_{,d}g^{cd}g^{ab}g_{ac,b}\end{pmatrix}\right)\cdot\frac{1}{2}g_{\alpha\beta}\\ +f_{,\alpha}\cdot f_{,\beta}\frac{2(n-1)(n-2)}{(n-3)^2(f+C_f)^2}\end{array}\right)\delta G^{\alpha\beta}\quad . \qquad (5)$$

While in other circumstances, like quantum gravity statistics [9, 10], we were interested in the case n→∞, leading to:

$$0=\left(\begin{array}{c}\left(R_{\alpha\beta}-\frac{2}{(f+C_f)}\left(f_{,\alpha\beta}-\frac{f_{,d}}{2}g^{cd}\left(g_{\alpha c,\beta}+g_{\beta c,\alpha}-g_{\alpha\beta,c}\right)\right)\right)+2\frac{f_{,\alpha}\cdot f_{,\beta}}{(f+C_f)^2}\\ -\left(R-\frac{2}{(f+C_f)}\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}-f_{,d}g^{cd}g^{ab}g_{ac,b}\right)\right)\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\end{array}\right)\delta G^{\alpha\beta}, \qquad (6)$$

We here concentrate on the special appearance of the number n=3, which would render the field equations to:

$$0=\left(R^{*}{}_{\alpha\beta}-R^{*}\frac{1}{2}G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}$$

$$=\left(\begin{array}{c}\left(R_{\alpha\beta}-\frac{C_f}{2}\left(\begin{array}{c} f_{,\alpha\beta}+f_{,ab}g_{\alpha\beta}g^{ab}+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-f_{,\alpha}g^{ab}g_{\beta b,a} \\ -f_{,\beta}g^{ab}g_{\alpha b,a}+f_{,d}g^{cd}\left(\begin{array}{c} g_{\alpha c,\beta}-\frac{3}{2}g_{\alpha c,\beta}-\frac{3}{2}g_{\beta c,\alpha} \\ +\frac{3}{2}g_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)\end{array}\right)\right) \\ -\left(R-\frac{C_f}{2}\left(\begin{array}{c}2\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right) \\ -3f_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right)\right)\cdot\frac{1}{2}g_{\alpha\beta}+\frac{C_f^2}{4}f_{,\alpha}\cdot f_{,\beta}\end{array}\right)\delta G^{\alpha\beta}. \qquad (7)$$

With only one non-linear term left. For completeness only we should remember that this can be expanded as follows:

$$\int\limits_V dx^n\sqrt{|G|}\cdot f_{,\alpha}\cdot f_{,\beta}\cdot\delta G^{\alpha\beta}=\int\limits_V dx^n\sqrt{|G|}\cdot\left(f_{,\alpha}\cdot f\right)_{,\beta}\cdot\delta G^{\alpha\beta}-\int\limits_V dx^n\sqrt{|G|}\cdot f_{,\alpha\beta}\cdot f\cdot\delta G^{\alpha\beta}. \qquad (8)$$

This leads to:

$$0=\left(R^{*}{}_{\alpha\beta}-R^{*}\frac{1}{2}G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}$$

$$=\left(\begin{array}{c}\left(R_{\alpha\beta}-\frac{C_f}{2}\left(\begin{array}{c} f_{,\alpha\beta}+f_{,ab}g_{\alpha\beta}g^{ab}+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-f_{,\alpha}g^{ab}g_{\beta b,a} \\ -f_{,\beta}g^{ab}g_{\alpha b,a}+f_{,d}g^{cd}\left(\begin{array}{c} g_{\alpha c,\beta}-\frac{3}{2}g_{\alpha c,\beta}-\frac{3}{2}g_{\beta c,\alpha} \\ +\frac{3}{2}g_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)\end{array}\right)\right) \\ -\left(R-\frac{C_f}{2}\left(\begin{array}{c}2\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right) \\ -3f_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right)\right)\cdot\frac{1}{2}g_{\alpha\beta} \\ +\frac{C_f^2}{4}\left(\left(f_{,\alpha}\cdot f\right)_{,\beta}-f_{,\alpha\beta}\cdot f\right)\end{array}\right)\delta G^{\alpha\beta}. \qquad (9)$$

It should also be pointed out – again, more or less only for completeness – that (7) is the kernel of a volume integral and that consequently the consideration of (8) could also be done as follows:

$$\int\limits_V dx^n\sqrt{|G|}\cdot f_{,\alpha}\cdot f_{,\beta}\cdot\delta G^{\alpha\beta}=\int\limits_V dx^n\sqrt{|G|}\cdot f_{,\alpha}\left(\delta G^{\alpha\beta}\right)f_{,\beta}. \qquad (10)$$

Now we find something similar to the quantum mechanical expectation value in the space with the metric $G_{\alpha\beta}$, and the wave function or “Führungsfeld” Ψ, which is defined as (see [9, 10]):

$$E\left[\hat{A}\right]=\int\limits_V dx^n\sqrt{|G|}\cdot\Psi^{*}\hat{A}\Psi. \qquad (11)$$

This connection will be discussed in one of our upcoming publications.

References

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