The Quantum Gravity Eikonal Equation

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1 Abstract

Starting from a scaled metric we were able to derive Quantum Gravity equations containing the vacuum Einstein-Field-Equations [A1, A2] for the metric tensor and the classical (relativity-compatible) Quantum Equations for the volume part of the metric [A3, A4, A5, A6, A7, A8]. This brings Quantum Theory and Einstein's General Theory of Relativity together.

When taking the resulting Quantum Gravity equations and assuming big numbers of degrees of freedom, we obtain the classical Eikonal or light-particle equation [A9], which is known for centuries. It describes the propagation of light rays in geometrical optics. Classically, it can be obtained from wave optics via a limiting procedures for wave numbers going to infinity. Interestingly, when applying the same technology directly onto our Quantum Gravity equations we do not obtain the classical Eikonal, but a slightly more complicated equation… still being similar to the classical one in character and sporting its non-linearity. Only certain additional conditions, like the assumption of a volume-restricted variation withing the Einstein-Hilbert action, gives the classical outcome.

The results of this paper lead us to a connection between the wave number of certain solutions for of a system and the latter's dimensionality.

1.1 Abstract References

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see also: https://lp.uni-goettingen.de/get/text/6216

2 Introduction

2.1 The Classical Eikonal Equation (e.g. [9])

Using the following approach for a wave function f:

$$f = f\left(A[x,y,\ldots]\cdot e^{L[x,y,\ldots]}\right) = A\left[a[x,y,\ldots]\right]\cdot e^{L\left[l[x,y,\ldots]\right]}, \tag{1}$$

and setting it in the following wave equation:

$$0 = \Delta f + k^2 b^2 f = g^{ab} f_{,ab} + \frac{1}{2} f_{,d} g^{cd} g^{ab} g_{ab,c} + f_{,d} g^{cd}{}_{,c} + k^2 b^2 f, \tag{2}$$

thereby using:

$$\begin{array}{c} f_{,a} = \left(A'\cdot a_{,a} + A\cdot L'\cdot l_{,a}\right)\cdot e^{L[l]} \\ f_{,d} = \left(A'\cdot a_{,d} + A\cdot L'\cdot l_{,d}\right)\cdot e^{L[l]} \\ f_{,ab} = \left(\begin{array}{c} \left(A'\cdot a_{,a} + A\cdot L'\cdot l_{,a}\right)\cdot e^{L[l]}\cdot L'\cdot l_{,b} \\ + \left(A''\cdot a_{,a} a_{,b} + A'\cdot a_{,ab} + A'\cdot a_{,b}\cdot L'\cdot l_{,a} + A\cdot L''\cdot l_{,a} l_{,b} + A\cdot L'\cdot l_{,ab}\right)\cdot e^{L[l]} \end{array}\right) \end{array}, \tag{3}$$

results in:

$$\begin{array}{l} 0 = \left(\begin{array}{c} g^{ab}\left(\begin{array}{c} \left(A'\cdot a_{,a} + A\cdot L'\cdot l_{,a}\right)\cdot e^{L[l]}\cdot L'\cdot l_{,b} \\ + \left(A''\cdot a_{,a} a_{,b} + A'\cdot a_{,ab} + A'\cdot a_{,b}\cdot L'\cdot l_{,a} + A\cdot L''\cdot l_{,a} l_{,b} + A\cdot L'\cdot l_{,ab}\right)\cdot e^{L[l]} \end{array}\right) \\ + \left(A'\cdot a_{,d} + A\cdot L'\cdot l_{,d}\right)\left(\frac{g^{cd} g^{ab} g_{ab,c}}{2} + g^{cd}{}_{,c}\right)\cdot e^{L[l]} + k^2 b^2 f \end{array}\right) \\ = \left(\begin{array}{c} g^{ab}\left(\begin{array}{c} \left(A'\cdot a_{,a} + i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)\cdot i\cdot k\cdot L'\cdot l_{,b} \\ + \left(\begin{array}{c} A''\cdot a_{,a} a_{,b} + A'\cdot a_{,ab} + A'\cdot a_{,b}\cdot i\cdot k\cdot L'\cdot l_{,a} \\ + A\cdot i\cdot k\cdot L''\cdot l_{,a} l_{,b} + A\cdot i\cdot k\cdot L'\cdot l_{,ab} \end{array}\right) \end{array}\right) \\ + \left(A'\cdot a_{,d} + A\cdot i\cdot k\cdot L'\cdot l_{,d}\right)\left(\frac{g^{cd} g^{ab} g_{ab,c}}{2} + g^{cd}{}_{,c}\right) + k^2 b^2 A \end{array}\right)\cdot e^{L[l]} \end{array}. \tag{4}$$

Making the exponent dependent on the wave number in the following way:

$$f = A\left[a[x,y,\ldots]\right]\cdot e^{i\cdot k\cdot L\left[l[x,y,\ldots]\right]}, \tag{5}$$

yields:

$$0 = \left(\begin{array}{c} g^{ab}\left(\begin{array}{c} \left(A'\cdot a_{,a} + i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)\cdot i\cdot k\cdot L'\cdot l_{,b} \\ + \left(\begin{array}{c} A''\cdot a_{,a} a_{,b} + A'\cdot a_{,ab} + A'\cdot a_{,b}\cdot i\cdot k\cdot L'\cdot l_{,a} \\ + A\cdot i\cdot k\cdot L''\cdot l_{,a} l_{,b} + A\cdot i\cdot k\cdot L'\cdot l_{,ab} \end{array}\right) \end{array}\right) \\ + \left(A'\cdot a_{,d} + A\cdot i\cdot k\cdot L'\cdot l_{,d}\right)\left(\frac{g^{cd} g^{ab} g_{ab,c}}{2} + g^{cd}{}_{,c}\right) + k^2 b^2 A \end{array}\right)\cdot e^{i\cdot k\cdot L[l]}. \tag{6}$$

For very large k the k-linear and the k² terms become dominant and we obtain:

$$
\begin{array}{c}
0=\left(\begin{array}{c}
g^{ab}\left(A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}\right)+\left(A'\cdot a_{,d}\right)\left(\frac{g^{cd}g^{ab}g_{ab,c}}{2}+g^{cd}{}_{,c}\right)\\
+g^{ab}\left(\begin{array}{c}A'\cdot a_{,a}\cdot i\cdot k\cdot L'\cdot l_{,b}+A'\cdot a_{,b}\cdot i\cdot k\cdot L'\cdot l_{,a}\\ +A\cdot i\cdot k\cdot L''\cdot l_{,a}l_{,b}+A\cdot i\cdot k\cdot L'\cdot l_{,ab}\end{array}\right)\\
+A\cdot i\cdot k\cdot L'\cdot l_{,d}\left(\frac{g^{cd}g^{ab}g_{ab,c}}{2}+g^{cd}{}_{,c}\right)\\
+g^{ab}\left(i\cdot k\cdot A\cdot L'\cdot l_{,a}\cdot i\cdot k\cdot L'\cdot l_{,b}\right)+k^2b^2A
\end{array}\right)\\
\xrightarrow{k\to"\infty"}\\
0=\left(\begin{array}{c}
i\cdot k\cdot g^{ab}\left(\begin{array}{c}A'\cdot a_{,a}\cdot L'\cdot l_{,b}+A'\cdot a_{,b}\cdot L'\cdot l_{,a}+A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\\ +A\cdot L'\cdot l_{,d}\left(\frac{g^{cd}g^{ab}g_{ab,c}}{2}+g^{cd}{}_{,c}\right)\end{array}\right)\\
+k^2A\left(b^2-g^{ab}\left(L'\right)^2\cdot l_{,a}l_{,b}\right)
\end{array}\right)
\end{array}
\quad . \qquad (7)
$$

Here we have with:

$$
0=b^2-g^{ab}\left(L'\right)^2\cdot l_{,a}l_{,b}\,, \qquad (8)
$$

the so-called Eikonal equation for l, the solution of which determines via:

$$
0=g^{ab}\left(\begin{array}{c}A'\cdot a_{,a}\cdot L'\cdot l_{,b}+A'\cdot a_{,b}\cdot L'\cdot l_{,a}+A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\\ +A\cdot L'\cdot l_{,d}\left(\frac{g^{cd}g^{ab}g_{ab,c}}{2}+g^{cd}{}_{,c}\right)\end{array}\right) \qquad (9)
$$

the functions A and a. The symmetry of the metric tensor allows us to simplify (9):

$$
\begin{array}{c}
0=g^{ab}\left(2A'\cdot a_{,a}\cdot L'\cdot l_{,b}+A\cdot\left(L''\cdot l_{,a}l_{,b}+L'\cdot l_{,ab}+L'\cdot l_{,d}\left(\frac{g^{cd}g^{ab}g_{ab,c}}{2}+g^{cd}{}_{,c}\right)\right)\right)\\
0=g^{ab}\left(L''\cdot l_{,a}l_{,b}+L'\cdot\left(2\left(\ln A\right)'\cdot a_{,a}\cdot l_{,b}+l_{,ab}+l_{,d}\left(\frac{g^{cd}g^{ab}g_{ab,c}}{2}+g^{cd}{}_{,c}\right)\right)\right)
\end{array}
\quad . \qquad (10)
$$

2.2 The “Theory of Everything”

We start with the following scaled metric tensor and force it into the Einstein-Hilbert action [1] variational problem as follows:

$$
G_{\alpha\beta}=g_{\alpha\beta}\cdot F[f] \quad\rightarrow\quad \delta W=0=\delta\int_V d^nx\sqrt{-G}\cdot R^* \qquad (11)
$$

Here G denotes the determinant of the metric tensor from (11) and R^* gives the corresponding Ricci scalar. Performing the variation with respect to the metric $G_{\alpha\beta}$ results in:

$$0=\left(\begin{array}{c}\boxed{R_{\alpha\beta}-R\frac{g_{\alpha\beta}}{2}}\\ -\frac{1}{2F}\left(\begin{array}{c}F_{,\alpha\beta}(n-2)+F_{,ab}g_{\alpha\beta}g^{ab}+F_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-\\ F_{,\alpha}g^{ab}g_{\beta b,a}-F_{,\beta}g^{ab}g_{\alpha b,a}+F_{,d}g^{cd}\left(\begin{array}{c}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)\end{array}\right)\\ +\frac{1}{4F^2}\left(F_{,\alpha}\cdot F_{,\beta}(3n-6)+g_{\alpha\beta}F_{,c}F_{,d}g^{cd}(4-n)\right)\\ +(n-1)\left(\frac{1}{2F}\left(\begin{array}{c}2\Delta F-2F_{,d}g^{cd}{}_{,c}\\ -\frac{n}{(n-1)}F_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right)+\frac{g^{ab}F_{,a}\cdot F_{,b}}{4F^2}(n-6)\right)\cdot\frac{g_{\alpha\beta}}{2}\end{array}\right)\delta G^{\alpha\beta} \tag{12}$$

and shows us that we have not only obtained the classical Einstein Theory of Relativity [2] (see boxed terms exactly giving the Einstein Field Equations in vacuum), but also a set of quantum field equations for the scaling function F, clearly playing the role of the wave-function. It was shown in our previous publications [3, 4, 5, 6, 7] that these additional terms are quantum equations fully covering the main aspects of relativistic classical quantum theory. So, we conclude, that we indeed have a Quantum Gravity Theory or "Theory of Everything", as one also calls it, at hand.

2.3 "Weak Gravity" and Linearity – The Transition to the Classical Quantum Theory

It was shown in [3, 5, 6, 7] that the so-called "weak gravity" condition:

$$\delta G^{\alpha\beta}=G^{\alpha\beta}\cdot\delta_0+\overbrace{G^{ab}\delta^{\alpha\beta}_{ab}}^{\text{Gravity}}\xrightarrow{\forall\,\delta^{\alpha\beta}_{ab}\ll\delta_0}=\frac{g^{\alpha\beta}}{F}\cdot\delta_0\,, \tag{13}$$

together with a setting for the scaling function F[f] as follows:

$$F[f]=\begin{cases}C_F\cdot\left(f+C_f\right)^{\frac{4}{n-2}} & n\neq 2\\ C_F\cdot e^{f\cdot C_f} & n=2\end{cases}. \tag{14}$$

leads to a significant simplification and scalarization of the quantum gravity field equations (12), namely:

$$0=R-\frac{F'}{2F}\left((n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)-nf_{,d}g^{cd}g^{ab}g_{ac,b}\right). \tag{15}$$

This equation is completely linear in f, which not only has the characteristics of a quantum wave function, but – for a change – gives us the opportunity to metrically see what QUANTUM actually means, namely, a volume jitter to the metric of the system in question… at least this is one quantum option, because we have already seen others, like the perturbated kernel (e.g. see [7]).

Interestingly, for metrics without shear elements:

$$g_{ij}=\begin{pmatrix}g_{00} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & g_{n-1n-1}\end{pmatrix};\quad g_{ii,i}=0\,, \tag{16}$$

and applying the solution for F[f] from (14) the derivative terms in (15), which is to say:

$$(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)-nf_{,d}g^{cd}g^{ab}g_{ac,b}\,. \tag{17}$$

converge to the ordinary Laplace operator, namely:

$$\begin{aligned}&R^{*}=0 \quad\rightarrow\quad 0=F\cdot R+F'\cdot(1-n)\cdot\Delta f\\ &\Rightarrow\quad 0=\begin{cases}\left(f-C_{f}\right)^{\frac{4}{n-2}}\cdot C_{F}\left(R+\dfrac{4}{n-2}\cdot\dfrac{(1-n)}{\left(f-C_{f}\right)}\cdot\Delta f\right) & n>2\\ e^{C_{f}\cdot f}\cdot C_{F}\left(R+C_{f}\cdot(1-n)\cdot\Delta f\right) & n=2\end{cases}\end{aligned} \tag{18}$$

We recognize the relativistic Klein-Gordon equation.

Thus, in the case of n>2 we always also have the option for a constant (broken symmetry) solution of the kind:

$$0=f-C_{f0}\quad\Rightarrow\quad f=C_{f0}\,. \tag{19}$$

In all other cases, meaning where $f\neq C_{f0}$, we have the simple equations:

$$0=\begin{cases}\left(f-C_{f0}\right)\cdot R+(1-n)\cdot\dfrac{4}{n-2}\cdot\Delta f & n>2\\ R+C_{f0}\cdot(1-n)\cdot\Delta f & n=2\end{cases}\,. \tag{20}$$

A critical argument should now be that this equation is not truly of Klein-Gordon character as it does not contain neither potential nor mass, but the first author of this paper has already shown that this problem is easily solved by adding additional dimensions carrying the right properties to produce masses and potentials due to entanglement (e.g. [3 – 7]).

3 Theory

3.1 Some General Considerations

3.1.1 Using the Scaling Function F[f]

Inserting a wrapping function F[f] and a bit of reordering of (12) leads to:

$$0=\left(R^{*}{}_{\alpha\beta}-\frac{1}{2}R^{*}G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}$$

$$=\begin{pmatrix}\left(R_{\alpha\beta}-\frac{F'}{2F}\begin{pmatrix}f_{,\alpha\beta}(n-2)+f_{,ab}g_{\alpha\beta}g^{ab}+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-f_{,\alpha}g^{ab}g_{\beta b,a}\\ -f_{,\beta}g^{ab}g_{\alpha b,a}+f_{,d}g^{cd}\begin{pmatrix}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{pmatrix}\end{pmatrix}\right)\\ -\left(R-\frac{F'}{2F}\begin{pmatrix}(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)\\ -nf_{,d}g^{cd}g^{ab}g_{ac,b}\end{pmatrix}\right)\cdot\frac{1}{2}g_{\alpha\beta}\\ +\frac{1}{4F^{2}}\begin{pmatrix}f_{,\alpha}\cdot f_{,\beta}(n-2)\left(3(F')^{2}-2FF''\right)\\ +g_{\alpha\beta}f_{,c}f_{,d}g^{cd}\begin{pmatrix}\left((F')^{2}(4-n)-2FF''\right)\\ +(n-1)\cdot\frac{1}{2}\left(4FF''+(F')^{2}(n-6)\right)\end{pmatrix}\end{pmatrix}\end{pmatrix}\delta G^{\alpha\beta}$$

$$=\begin{pmatrix}\left(R_{\alpha\beta}-\frac{F'}{2F}\begin{pmatrix}f_{,\alpha\beta}(n-2)+f_{,ab}g_{\alpha\beta}g^{ab}+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-f_{,\alpha}g^{ab}g_{\beta b,a}\\ -f_{,\beta}g^{ab}g_{\alpha b,a}+f_{,d}g^{cd}\begin{pmatrix}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{pmatrix}\end{pmatrix}\right)\\ -\left(R-\frac{F'}{2F}\begin{pmatrix}(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)\\ -nf_{,d}g^{cd}g^{ab}g_{ac,b}\end{pmatrix}\right)\cdot\frac{1}{2}g_{\alpha\beta}\\ +\frac{n-2}{4F^{2}}\left(f_{,\alpha}\cdot f_{,\beta}\left(3(F')^{2}-2FF''\right)+g_{\alpha\beta}f_{,c}f_{,d}g^{cd}\left((F')^{2}\cdot\frac{1}{2}(n-7)+2FF''\right)\right)\end{pmatrix}\delta G^{\alpha\beta}\quad. \tag{21}$$

We recognize two non-linear terms in the last line. With the setting

$$F[f]=C_{f1}\left(f+C_{f0}\right)^{-2}, \tag{22}$$

we can get rid of one of them, resulting in:

$$
0=\left(\begin{array}{c}
\left(f+C_{f0}\right)^2\cdot\left(g_{\alpha\beta}R\cdot\left(\frac{1}{2}+H\right)-R_{\alpha\beta}\right)\\
\left(f+C_{f0}\right)\cdot\left(\begin{array}{c}
-\left(\begin{array}{c}
f_{,\alpha\beta}(n-2)+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-f_{,\alpha}g^{ab}g_{\beta b,a}\\
-f_{,\beta}g^{ab}g_{\alpha b,a}+f_{,d}g^{cd}\left(g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}+\frac{1}{2}ng_{\alpha\beta,c}\right)
\end{array}\right)\\
-g_{\alpha\beta}\left(f_{,ab}g^{ab}+f_{,d}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right)\\
+\frac{g_{\alpha\beta}}{2}\left(\begin{array}{c}
(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)\\
-nf_{,d}g^{cd}g^{ab}g_{ac,b}
\end{array}\right)
\end{array}\right)\\
-g_{\alpha\beta}f_{,c}f_{,d}g^{cd}\left(\frac{(n-1)(n-2)}{2}\right)
\end{array}\right). \tag{23}
$$

Interested in getting rid of the second non-linear (scalar) term in (21), we have to set:

$$
F[f]=\begin{cases} C_F\cdot\left(f+C_f\right)^{\frac{4}{(n-3)}} & n\neq 3\\ C_F\cdot e^{f\cdot C_f} & n=3\end{cases}. \tag{24}
$$

In this case (21) yields the following field equations:

$$
\begin{aligned}
0 &= \left(R^{*}{}_{\alpha\beta} - \frac{R^{*}}{2} G_{\alpha\beta}\right) \overbrace{\left(\frac{1}{F}\cdot \delta g^{\alpha\beta} + g^{\alpha\beta}\cdot \delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}} \\
&= \left(
\begin{matrix}
\left(R_{\alpha\beta} - \dfrac{F'}{2F}\left(
\begin{matrix}
f_{,\alpha\beta}(n-2) + f_{,ab} g_{\alpha\beta} g^{ab} + f_{,a} g^{ab}\left(g_{\beta b,\alpha} - g_{\beta\alpha,b}\right) - f_{,\alpha} g^{ab} g_{\beta b,a} \\
-f_{,\beta} g^{ab} g_{\alpha b,a} + f_{,d} g^{cd}\left(
\begin{matrix}
g_{\alpha c,\beta} - \frac{1}{2} n g_{\alpha c,\beta} - \frac{1}{2} n g_{\beta c,\alpha} \\
+\frac{1}{2} n g_{\alpha\beta,c} + \frac{1}{2} g_{\alpha\beta} g_{ab,c} g^{ab}
\end{matrix}\right)
\end{matrix}\right)\right) \\
-\left(R - \dfrac{F'}{4F}\left(
\begin{matrix}
(n-1)\left(2g^{ab} f_{,ab} + f_{,d} g^{cd} g^{ab} g_{ab,c}\right) \\
-n f_{,d} g^{cd} g^{ab} g_{ac,b}
\end{matrix}\right)\right)\cdot g_{\alpha\beta} \\
+\dfrac{1}{4F^{2}} f_{,\alpha}\cdot f_{,\beta}(n-2)\left(3(F')^{2} - 2FF''\right)
\end{matrix}
\right)\delta G^{\alpha\beta} \\
&= \left(
\begin{matrix}
\left(
\begin{matrix}
R_{\alpha\beta} - \dfrac{2(n-2)}{(n-2)(n-3)(f+C_f)} \\
\times\left(
\begin{matrix}
f_{,\alpha\beta}(n-2) + f_{,ab} g_{\alpha\beta} g^{ab} + f_{,a} g^{ab}\left(g_{\beta b,\alpha} - g_{\beta\alpha,b}\right) - f_{,\alpha} g^{ab} g_{\beta b,a} \\
-f_{,\beta} g^{ab} g_{\alpha b,a} + f_{,d} g^{cd}\left(
\begin{matrix}
g_{\alpha c,\beta} - \frac{1}{2} n g_{\alpha c,\beta} - \frac{1}{2} n g_{\beta c,\alpha} \\
+\frac{1}{2} n g_{\alpha\beta,c} + \frac{1}{2} g_{\alpha\beta} g_{ab,c} g^{ab}
\end{matrix}\right)
\end{matrix}\right)
\end{matrix}\right) \\
-\left(
\begin{matrix}
R - \dfrac{2(n-2)}{(n-2)(n-3)(f+C_f)} \\
\times\left(
\begin{matrix}
(n-1)\left(2g^{ab} f_{,ab} + f_{,d} g^{cd} g^{ab} g_{ab,c}\right) \\
-n f_{,d} g^{cd} g^{ab} g_{ac,b}
\end{matrix}\right)
\end{matrix}\right)\cdot \dfrac{g_{\alpha\beta}}{2} + f_{,\alpha}\cdot f_{,\beta} \dfrac{2(n-1)(n-2)}{(n-3)^{2}(f+C_f)^{2}}
\end{matrix}
\right)\delta G^{\alpha\beta} \quad . \qquad (25)
\end{aligned}
$$

3.1.2 The Wave Approach

Starting with the simple case of metrics of constants, we obtain the following from the general quantum gravity equation (12):

$$
\begin{aligned}
0 &= \left(\begin{array}{c} -\dfrac{1}{2F}\left(F_{,\alpha\beta}(n-2) + F_{,ab} g_{\alpha\beta} g^{ab}\right) \\ +\dfrac{1}{4F^2}\left(F_{,\alpha}\cdot F_{,\beta}(3n-6) + g_{\alpha\beta} F_{,c} F_{,d} g^{cd}(4-n)\right) \\ +(n-1)\left(\dfrac{\Delta F}{F} + \dfrac{g^{ab} F_{,a}\cdot F_{,b}}{4F^2}(n-6)\right)\cdot \dfrac{g_{\alpha\beta}}{2} \end{array} \right) \delta G^{\alpha\beta} \\
&= \left(\begin{array}{c} -\left(F_{,\alpha\beta}(n-2) + F_{,ab} g_{\alpha\beta} g^{ab}\right) \\ +\dfrac{1}{2F}\left(F_{,\alpha}\cdot F_{,\beta}(3n-6) + g_{\alpha\beta} F_{,c} F_{,d} g^{cd}(4-n)\right) \\ +(n-1)\left(F_{,ab} + \dfrac{F_{,a}\cdot F_{,b}}{4F}(n-6)\right)\cdot g^{ab} g_{\alpha\beta} \end{array} \right) \frac{\delta G^{\alpha\beta}}{2F} \quad . \\
&= (n-2)\left(\begin{array}{c} F_{,ab} g_{\alpha\beta} g^{ab} - F_{,\alpha\beta} \\ +\dfrac{1}{2F}\left(3F_{,\alpha}\cdot F_{,\beta} + g_{\alpha\beta} F_{,c} F_{,d} g^{cd}\dfrac{(n-7)}{2}\right) \end{array} \right) \frac{\delta G^{\alpha\beta}}{2F}
\end{aligned} \tag{26}
$$

In some cases it might be good to have the inner terms ordered with respect to the true tensor and the scalar part (the latter multiplied with the metric tensor $g_{\alpha\beta}$):

$$
0 = (n-2)\left(\frac{3F_{,\alpha}\cdot F_{,\beta}}{2F} - F_{,\alpha\beta} + g_{\alpha\beta} g^{ab}\left(F_{,ab} + \frac{F_{,a}F_{,b}}{4F}(n-7) \right) \right) \frac{\delta G^{\alpha\beta}}{2F} . \tag{27}
$$

Now we assume the following type of solution:

$$
F = F(f) = F\left(A[x,y,\ldots]\cdot e^{L[x,y,\ldots]}\right) = A\left[a[x,y,\ldots]\right]\cdot e^{L\left[l[x,y,\ldots]\right]} . \tag{28}
$$

The corresponding derivatives read:

$$
\begin{aligned}
F_{,\alpha} &= \left(A'\cdot a_{,\alpha} + A\cdot L'\cdot l_{,\alpha}\right)\cdot e^{L[l]} \\
F_{,\beta} &= \left(A'\cdot a_{,\beta} + A\cdot L'\cdot l_{,\beta}\right)\cdot e^{L[l]}
\end{aligned} \quad , \tag{29}
$$

$$
\begin{aligned}
F_{,\alpha\beta} &= \left(\begin{array}{c} \left(A'\cdot a_{,\alpha} + A\cdot L'\cdot l_{,\alpha}\right)\cdot e^{L[l]}\cdot L'\cdot l_{,\beta} \\ +\left(A''\cdot a_{,\alpha} a_{,\beta} + A'\cdot a_{,\alpha\beta} + A'\cdot a_{,\beta}\cdot L'\cdot l_{,\alpha} + A\cdot L''\cdot l_{,\alpha} l_{,\beta} + A\cdot L'\cdot l_{,\alpha\beta}\right)\cdot e^{L[l]} \end{array} \right) \\
F_{,a} &= \left(A'\cdot a_{,a} + A\cdot L'\cdot l_{,a}\right)\cdot e^{L[l]} \\
F_{,b} &= \left(A'\cdot a_{,b} + A\cdot L'\cdot l_{,b}\right)\cdot e^{L[l]} \qquad . \\
F_{,ab} &= \left(\begin{array}{c} \left(A'\cdot a_{,a} + A\cdot L'\cdot l_{,a}\right)\cdot e^{L[l]}\cdot L'\cdot l_{,b} \\ +\left(A''\cdot a_{,a} a_{,b} + A'\cdot a_{,ab} + A'\cdot a_{,b}\cdot L'\cdot l_{,a} + A\cdot L''\cdot l_{,a} l_{,b} + A\cdot L'\cdot l_{,ab}\right)\cdot e^{L[l]} \end{array} \right)
\end{aligned} \tag{30}
$$

This makes (27) to result in:

$$0=\left(\begin{array}{c} e^{2L[l]}\dfrac{3\left(A'\cdot a_{,\alpha}+A\cdot L'\cdot l_{,\alpha}\right)\cdot\left(A'\cdot a_{,\beta}+A\cdot L'\cdot l_{,\beta}\right)}{2F} \\ -\left(\begin{array}{c}\left(A'\cdot a_{,\alpha}+A\cdot L'\cdot l_{,\alpha}\right)\cdot e^{L[l]}\cdot L'\cdot l_{,\beta} \\ +\left(\begin{array}{c}A''\cdot a_{,\alpha}a_{,\beta}+A'\cdot a_{,\alpha\beta}+A'\cdot a_{,\beta}\cdot L'\cdot l_{,\alpha} \\ +A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot L'\cdot l_{,\alpha\beta}\end{array}\right)\cdot e^{L[l]}\end{array}\right) \\ +g_{\alpha\beta}g^{ab}\left(\begin{array}{c}\left(\begin{array}{c}\left(A'\cdot a_{,a}+A\cdot L'\cdot l_{,a}\right)\cdot e^{L[l]}\cdot L'\cdot l_{,b} \\ +\left(\begin{array}{c}A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}+A'\cdot a_{,b}\cdot L'\cdot l_{,a} \\ +A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\end{array}\right)\cdot e^{L[l]}\end{array}\right) \\ +\dfrac{\left(A'\cdot a_{,a}+A\cdot L'\cdot l_{,a}\right)\cdot\left(A'\cdot a_{,b}+A\cdot L'\cdot l_{,b}\right)}{4F}e^{2L[l]}(n-7)\end{array}\right)\end{array}\right). \tag{31}$$

Now we demand a[x,y,…]=l[x,y,…], which yields:

$$0=\left(\begin{array}{c} e^{2L[l]}\dfrac{3\left(A'\cdot l_{,\alpha}+A\cdot L'\cdot l_{,\alpha}\right)\cdot\left(A'\cdot l_{,\beta}+A\cdot L'\cdot l_{,\beta}\right)}{2F} \\ -\left(\begin{array}{c}\left(A'\cdot l_{,\alpha}+A\cdot L'\cdot l_{,\alpha}\right)\cdot e^{L[l]}\cdot L'\cdot l_{,\beta} \\ +\left(\begin{array}{c}A''\cdot l_{,\alpha}l_{,\beta}+A'\cdot l_{,\alpha\beta}+A'\cdot l_{,\beta}\cdot L'\cdot l_{,\alpha} \\ +A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot L'\cdot l_{,\alpha\beta}\end{array}\right)\cdot e^{L[l]}\end{array}\right) \\ +g_{\alpha\beta}g^{ab}\left(\begin{array}{c}\left(\begin{array}{c}\left(A'\cdot l_{,a}+A\cdot L'\cdot l_{,b}\right)\cdot e^{L[l]}\cdot L'\cdot l_{,b} \\ +\left(\begin{array}{c}A''\cdot l_{,a}l_{,b}+A'\cdot l_{,ab}+A'\cdot l_{,b}\cdot L'\cdot l_{,a} \\ +A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\end{array}\right)\cdot e^{L[l]}\end{array}\right) \\ +\dfrac{\left(A'\cdot l_{,a}+A\cdot L'\cdot l_{,a}\right)\cdot\left(A'\cdot l_{,b}+A\cdot L'\cdot l_{,b}\right)}{4F}(n-7)\end{array}\right)\end{array}\right) \tag{32}$$

and can be simplified to:

$$0=\begin{pmatrix} e^{2L[l]}\dfrac{3\left((A')^2+2A\cdot A'L'+A^2\cdot(L')^2\right)}{2F}\cdot l_{,\alpha}l_{,\beta} \\ -\begin{pmatrix} (A'+A\cdot L')\cdot e^{L[l]}\cdot L'\cdot l_{,\alpha}l_{,\beta} \\ +\begin{pmatrix} A''\cdot l_{,\alpha}l_{,\beta}+A'\cdot l_{,\alpha\beta}+A'\cdot L'\cdot l_{,\alpha}l_{,\beta} \\ +A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot L'\cdot l_{,\alpha\beta}\end{pmatrix}\cdot e^{L[l]}\end{pmatrix} \\ +g_{\alpha\beta}g^{ab}\begin{pmatrix}\begin{pmatrix} (A'+A\cdot L')\cdot e^{L[l]}\cdot L'\cdot l_{,a}l_{,b} \\ +\begin{pmatrix} A''\cdot l_{,a}l_{,b}+A'\cdot l_{,ab}+A'\cdot L'\cdot l_{,a}l_{,b} \\ +A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\end{pmatrix}\cdot e^{L[l]}\end{pmatrix} \\ +e^{2L[l]}\dfrac{\left((A')^2+2A\cdot A'L'+A^2\cdot(L')^2\right)}{4F}\cdot l_{,a}l_{,b}(n-7)\end{pmatrix}\end{pmatrix}$$

$$=\begin{pmatrix} g_{\alpha\beta}g^{ab}l_{,ab}(A'+A\cdot L')-l_{,\alpha\beta}(A'+A\cdot L') \\ +\begin{pmatrix} e^{L[l]}\dfrac{3\left((A')^2+2A\cdot A'L'+A^2\cdot(L')^2\right)}{2F} \\ -\left((A'+A\cdot L')L'+A''+A'\cdot L'+A\cdot L''\right)\end{pmatrix}\cdot l_{,\alpha}l_{,\beta} \\ +g_{\alpha\beta}g^{ab}\cdot l_{,a}l_{,b}\begin{pmatrix}(A'+A\cdot L')\cdot L'+A''+A'\cdot L'+A\cdot L'' \\ +e^{L[l]}\dfrac{\left((A')^2+2A\cdot A'L'+A^2\cdot(L')^2\right)}{4F}(n-7)\end{pmatrix}\end{pmatrix}\cdot e^{L[l]}\,. \tag{33}$$

We see, that with the – apparently comfortable - settings:

$$\begin{aligned}0&=(A')^2+2A\cdot A'L'+A^2\cdot(L')^2\\ 0&=(A'+A\cdot L')L'+A''+A'\cdot L'+A\cdot L''\end{aligned}\,. \tag{34}$$

we would obtain a linear differential equation in l:

$$\begin{aligned}0&=\left(g_{\alpha\beta}g^{ab}l_{,ab}(A'+A\cdot L')-l_{,\alpha\beta}(A'+A\cdot L')\right)\cdot e^{L[l]}\\ &=\left(g_{\alpha\beta}g^{ab}l_{,ab}-l_{,\alpha\beta}\right)\cdot(A'+A\cdot L')\cdot e^{L[l]}\end{aligned}\,. \tag{35}$$

Unfortunately, the only solution we can find is the trivial one with F being a constant via:

$$A\left[a[x,y,\ldots]\right]=C_A\cdot e^{-L\left[l[x,y,\ldots]\right]}\quad\Rightarrow\quad F=F(f)=\text{const}\,. \tag{36}$$

Going back to (33) and incorporating F, we can reshape the last line as follows:

$$
\begin{aligned}
0 &= \left(\begin{array}{c}
g_{\alpha\beta} g^{ab} l_{,ab} \left(A' + A \cdot L' \right) - l_{,\alpha\beta} \left(A' + A \cdot L' \right) \\
+ \left(\begin{array}{c}
e^{L[l]} \dfrac{3\left((A')^2 + 2A \cdot A'L' + A^2 \cdot (L')^2 \right)}{2A[a] \cdot e^{L[l]}} \\
-\left((A' + A \cdot L') L' + A'' + A' \cdot L' + A \cdot L'' \right)
\end{array} \right) \cdot l_{,\alpha} l_{,\beta} \\
+ g_{\alpha\beta} g^{ab} \cdot l_{,a} l_{,b} \left(\begin{array}{c}
(A' + A \cdot L') \cdot L' + A'' + A' \cdot L' + A \cdot L'' \\
+ e^{L[l]} \dfrac{\left((A')^2 + 2A \cdot A'L' + A^2 \cdot (L')^2 \right)}{4A[a] \cdot e^{L[l]}} (n-7)
\end{array} \right)
\end{array} \right) \cdot e^{L[l]} \\
&= \left(\begin{array}{c}
\left(g_{\alpha\beta} g^{ab} l_{,ab} - l_{,\alpha\beta} \right) \left(A' + A \cdot L' \right) \\
+ \left(\begin{array}{c}
\dfrac{3\left((A')^2 + 2A \cdot A'L' + A^2 \cdot (L')^2 \right)}{2A} \\
-\left((A' + A \cdot L') L' + A'' + A' \cdot L' + A \cdot L'' \right)
\end{array} \right) \cdot l_{,\alpha} l_{,\beta} \\
+ g_{\alpha\beta} g^{ab} \cdot l_{,a} l_{,b} \left(\begin{array}{c}
(A' + A \cdot L') \cdot L' + A'' + A' \cdot L' + A \cdot L'' \\
+ \dfrac{\left((A')^2 + 2A \cdot A'L' + A^2 \cdot (L')^2 \right)}{4A} (n-7)
\end{array} \right)
\end{array} \right) \cdot e^{L[l]}
\end{aligned} \qquad (37)
$$

With the two conditions:

$$
0 = \left(\begin{array}{c}
\dfrac{3\left((A')^2 + 2A \cdot A'L' + A^2 \cdot (L')^2 \right)}{2A} \\
-\left((A' + A \cdot L') L' + A'' + A' \cdot L' + A \cdot L'' \right)
\end{array} \right), \qquad (38)
$$

$$
0 = \left(\begin{array}{c}
(A' + A \cdot L') \cdot L' + A'' + A' \cdot L' + A \cdot L'' \\
+ \dfrac{\left((A')^2 + 2A \cdot A'L' + A^2 \cdot (L')^2 \right)}{4A} (n-7)
\end{array} \right), \qquad (39)
$$

we would avoid the non-linearities, ending up with (35) again. The only solution we can find here, however, is the ones we already know (e.g. [3 -7]) from our simple wrapping function F=F[f], now with (c.f. (22) and (24)):

$$
A\left[l[x, y, \ldots] \right] = C_{f1} \left(l[x, y, \ldots] + C_{f0} \right)^{-2} \cdot e^{-L\left[l[x,y,\ldots] \right]}, \qquad (40)
$$

$$
A\left[l[x, y, \ldots] \right] = e^{-L\left[l[x,y,\ldots] \right]} \begin{cases} C_F \cdot \left(l[x, y, \ldots] + C_f \right)^{\frac{4}{(n-3)}} & n \neq 3 \\ C_F \cdot e^{l[x,y,\ldots] \cdot C_f} & n = 3 \end{cases}. \qquad (41)
$$

Going back to (31), not demanding a[x,y,…]=l[x,y,…], results in:

$$0=\left(\begin{array}{c}\dfrac{3\left(A'\cdot a_{,\alpha}+A\cdot L'\cdot l_{,\alpha}\right)\cdot\left(A'\cdot a_{,\beta}+A\cdot L'\cdot l_{,\beta}\right)}{2A}\\ -\left(\begin{array}{c}\left(A'\cdot a_{,\alpha}+A\cdot L'\cdot l_{,\alpha}\right)\cdot L'\cdot l_{,\beta}\\ +\left(\begin{array}{c}A''\cdot a_{,\alpha}a_{,\beta}+A'\cdot a_{,\alpha\beta}+A'\cdot a_{,\beta}\cdot L'\cdot l_{,\alpha}\\ +A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot L'\cdot l_{,\alpha\beta}\end{array}\right)\end{array}\right)\\ +g_{\alpha\beta}g^{ab}\left(\begin{array}{c}\left(\begin{array}{c}\left(A'\cdot a_{,a}+A\cdot L'\cdot l_{,a}\right)\cdot L'\cdot l_{,b}\\ +\left(\begin{array}{c}A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}+A'\cdot a_{,b}\cdot L'\cdot l_{,a}\\ +A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\end{array}\right)\end{array}\right)\\ +\dfrac{\left(A'\cdot a_{,a}+A\cdot L'\cdot l_{,a}\right)\cdot\left(A'\cdot a_{,b}+A\cdot L'\cdot l_{,b}\right)}{4A}(n-7)\end{array}\right)\end{array}\right)\cdot e^{L[l]}. \quad (42)$$

Aiming for the Quantum Gravity equivalent of the Eikonal equation, we adjust F as follows:

$$F=F(f)=F\left(A\left[x,y,\ldots\right]\cdot e^{L[x,y,\ldots]}\right)=A\left[a\left[x,y,\ldots\right]\right]\cdot e^{i\cdot k\cdot L\left[l[x,y,\ldots]\right]}, \quad (43)$$

with:

$$\begin{array}{c}F_{,\alpha}=\left(A'\cdot a_{,\alpha}+i\cdot k\cdot A\cdot L'\cdot l_{,\alpha}\right)\cdot e^{i\cdot k\cdot L[l]}\\ F_{,\beta}=\left(A'\cdot a_{,\beta}+i\cdot k\cdot A\cdot L'\cdot l_{,\beta}\right)\cdot e^{i\cdot k\cdot L[l]}\\ F_{,\alpha\beta}=\left(\begin{array}{c}\left(A'\cdot a_{,\alpha}+i\cdot k\cdot A\cdot L'\cdot l_{,\alpha}\right)\cdot i\cdot k\cdot L'\cdot l_{,\beta}\\ +A''\cdot a_{,\alpha}a_{,\beta}+A'\cdot a_{,\alpha\beta}+i\cdot k\cdot\left(A'\cdot L'\cdot a_{,\beta}\cdot l_{,\alpha}+A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot L'\cdot l_{,\alpha\beta}\right)\end{array}\right)\cdot e^{i\cdot k\cdot L[l]}\\ F_{,a}=\left(A'\cdot a_{,a}+i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)\cdot e^{i\cdot k\cdot L[l]}\\ F_{,b}=\left(A'\cdot a_{,b}+i\cdot k\cdot A\cdot L'\cdot l_{,b}\right)\cdot e^{i\cdot k\cdot L[l]}\\ F_{,ab}=\left(\begin{array}{c}\left(A'\cdot a_{,a}+i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)\cdot i\cdot k\cdot L'\cdot l_{,b}\\ +A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}+i\cdot k\cdot\left(A'\cdot a_{,b}\cdot L'\cdot l_{,a}+A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\right)\end{array}\right)\cdot e^{i\cdot k\cdot L[l]}\end{array}, \quad (44)$$

thereby changing (42) as follows:

$$0=\left(\begin{array}{c}\dfrac{3\left(A'\cdot a_{,\alpha}+i\cdot k\cdot A\cdot L'\cdot l_{,\alpha}\right)\cdot\left(A'\cdot a_{,\beta}+i\cdot k\cdot A\cdot L'\cdot l_{,\beta}\right)}{2A}\\ -\left(\begin{array}{c}\left(A'\cdot a_{,\alpha}+i\cdot k\cdot A\cdot L'\cdot l_{,\alpha}\right)\cdot i\cdot k\cdot L'\cdot l_{,\beta}\\ +\left(\begin{array}{c}A''\cdot a_{,\alpha}a_{,\beta}+A'\cdot a_{,\alpha\beta}+A'\cdot a_{,\beta}\cdot i\cdot k\cdot L'\cdot l_{,\alpha}\\ +i\cdot k\cdot A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot i\cdot k\cdot L'\cdot l_{,\alpha\beta}\end{array}\right)\end{array}\right)\\ +g_{\alpha\beta}g^{ab}\left(\begin{array}{c}\left(\begin{array}{c}\left(A'\cdot a_{,a}+i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)\cdot i\cdot k\cdot L'\cdot l_{,b}\\ +\left(\begin{array}{c}A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}+A'\cdot a_{,b}\cdot i\cdot k\cdot L'\cdot l_{,a}\\ +i\cdot k\cdot A\cdot L''\cdot l_{,a}l_{,b}+A\cdot i\cdot k\cdot L'\cdot l_{,ab}\end{array}\right)\end{array}\right)\\ +\dfrac{\left(A'\cdot a_{,a}+i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)\cdot\left(A'\cdot a_{,b}+i\cdot k\cdot A\cdot L'\cdot l_{,b}\right)}{4A}(n-7)\end{array}\right)\end{array}\right)\cdot e^{i\cdot k\cdot L[l]}$$

$$\xrightarrow{k\to\text{"}\infty\text{"}}$$

$$0=\left(\begin{array}{c}-k^2\dfrac{3A^2\cdot(L')^2\cdot l_{,\alpha}\cdot l_{,\beta}}{2A}+k^2A\cdot(L')^2\cdot l_{,\alpha}\cdot l_{,\beta}\\ +g_{\alpha\beta}g^{ab}\left(-k^2\dfrac{A^2\cdot(L')^2\cdot l_{,a}\cdot l_{,b}}{4A}(n-7)-k^2A\cdot(L')^2\cdot l_{,a}\cdot l_{,b}\right)\end{array}\right)\cdot e^{i\cdot k\cdot L[l]}$$
$$=-\frac{k^2A\cdot(L')^2}{2}\cdot\left(l_{,\alpha}\cdot l_{,\beta}+\frac{l_{,a}\cdot l_{,b}g_{\alpha\beta}g^{ab}}{2}(n-3)\right)\cdot e^{i\cdot k\cdot L[l]}\quad. \qquad (45)$$

The remaining equation would then read:

$$0=\left(\begin{array}{c}\dfrac{3A'\cdot a_{,\alpha}A'\cdot a_{,\beta}}{2A}-A''\cdot a_{,\alpha}a_{,\beta}-A'\cdot a_{,\alpha\beta}+g_{\alpha\beta}g^{ab}\left(\begin{array}{c}A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}\\ +A'\cdot\dfrac{\left(A'\cdot a_{,a}a_{,b}\right)}{4A}(n-7)\end{array}\right)\\ +i\cdot k\cdot\left(\begin{array}{c}\dfrac{3A'\cdot L'\left(a_{,\alpha}l_{,\beta}+l_{,\alpha}a_{,\beta}\right)}{2}-\left(\begin{array}{c}A'\cdot a_{,\alpha}\cdot L'\cdot l_{,\beta}+A'\cdot a_{,\beta}\cdot L'\cdot l_{,\alpha}\\ +A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot L'\cdot l_{,\alpha\beta}\end{array}\right)\\ +g_{\alpha\beta}g^{ab}\left(\begin{array}{c}A'\cdot a_{,a}\cdot L'\cdot l_{,b}+A'\cdot a_{,b}\cdot L'\cdot l_{,a}+A\cdot L''\cdot l_{,a}l_{,b}\\ +A\cdot L'\cdot l_{,ab}+A'\cdot\dfrac{\left(a_{,a}L'\cdot l_{,b}+L'\cdot l_{,a}a_{,b}\right)}{4}(n-7)\end{array}\right)\end{array}\right)\end{array}\right). \qquad (46)$$

We recognize the complementary to the classical Eikonal “side-kick” (9) to be:

$$0=\left(\begin{array}{c}\dfrac{3A'\cdot L'\left(a_{,\alpha}l_{,\beta}+l_{,\alpha}a_{,\beta}\right)}{2}-\left(\begin{array}{c}A'\cdot a_{,\alpha}\cdot L'\cdot l_{,\beta}+A'\cdot a_{,\beta}\cdot L'\cdot l_{,\alpha}\\ +A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot L'\cdot l_{,\alpha\beta}\end{array}\right)\\ +g_{\alpha\beta}g^{ab}\left(\begin{array}{c}A'\cdot a_{,a}\cdot L'\cdot l_{,b}+A'\cdot a_{,b}\cdot L'\cdot l_{,a}+A\cdot L''\cdot l_{,a}l_{,b}\\ +A\cdot L'\cdot l_{,ab}+A'\cdot\dfrac{\left(a_{,a}L'\cdot l_{,b}+L'\cdot l_{,a}a_{,b}\right)}{4}(n-7)\end{array}\right)\end{array}\right). \qquad (47)$$

3.1.2.1 The General Case

For completeness we now consider the general metric case, leading to:

$$
\begin{aligned}
0=\Bigg(&\boxed{R_{\alpha\beta}-R\frac{g_{\alpha\beta}}{2}}\\
&-\frac{1}{2A}\left(\begin{array}{c}
\left(\begin{array}{c}
\left(A'\cdot a_{,\alpha}+i\cdot k\cdot A\cdot L'\cdot l_{,\alpha}\right)\cdot i\cdot k\cdot L'\cdot l_{,\beta}\\
+A''\cdot a_{,\alpha}a_{,\beta}+A'\cdot a_{,\alpha\beta}\\
+i\cdot k\cdot\left(A'\cdot L'\cdot a_{,\beta}\cdot l_{,\alpha}+A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot L'\cdot l_{,\alpha\beta}\right)
\end{array}\right)(n-2)\\
+\left(\begin{array}{c}
\left(A'\cdot a_{,a}+i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)\cdot i\cdot k\cdot L'\cdot l_{,b}\\
+A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}\\
+i\cdot k\cdot\left(A'\cdot a_{,b}\cdot L'\cdot l_{,a}+A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\right)
\end{array}\right)g_{\alpha\beta}g^{ab}\\
+\left(A'\cdot a_{,a}+i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-\\
\left(A'\cdot a_{,\alpha}+i\cdot k\cdot A\cdot L'\cdot l_{,\alpha}\right)g^{ab}g_{\beta b,a}\\
-\left(A'\cdot a_{,\beta}+i\cdot k\cdot A\cdot L'\cdot l_{,\beta}\right)g^{ab}g_{\alpha b,a}\\
+\left(A'\cdot a_{,d}+i\cdot k\cdot A\cdot L'\cdot l_{,d}\right)g^{cd}\left(\begin{array}{c}
g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\
+\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}
\end{array}\right)
\end{array}\right)\\
&+\frac{1}{4A^2}\left(\begin{array}{c}
\left(A'\cdot a_{,\alpha}+i\cdot k\cdot A\cdot L'\cdot l_{,\alpha}\right)\cdot\left(A'\cdot a_{,\beta}+i\cdot k\cdot A\cdot L'\cdot l_{,\beta}\right)(3n-6)\\
+g_{\alpha\beta}\left(A'\cdot a_{,a}+i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)\cdot\left(A'\cdot a_{,b}+i\cdot k\cdot A\cdot L'\cdot l_{,b}\right)g^{ab}(4-n)
\end{array}\right)\\
&+(n-1)\left(\begin{array}{c}
\frac{1}{2A}\left(\begin{array}{c}
2g^{ab}\left(\begin{array}{c}
\left(A'\cdot a_{,a}+i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)\cdot i\cdot k\cdot L'\cdot l_{,b}\\
+A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}\\
+i\cdot k\cdot\left(A'\cdot a_{,b}\cdot L'\cdot l_{,a}+A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\right)
\end{array}\right)\\
+\left(A'\cdot a_{,d}+i\cdot k\cdot A\cdot L'\cdot l_{,d}\right)\left(g^{cd}g^{ab}g_{ab,c}-\frac{ng^{cd}g^{ab}g_{ac,b}}{(n-1)}\right)
\end{array}\right)\\
+\frac{g^{ab}\left(A'\cdot a_{,a}+i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)\left(A'\cdot a_{,b}+i\cdot k\cdot A\cdot L'\cdot l_{,b}\right)}{4A^2}(n-6)
\end{array}\right)\frac{g_{\alpha\beta}}{2}\Bigg)
\end{aligned}
\quad . \qquad (48)
$$

Ordering with respect to the powers of k yields:

$$
0=\left(\begin{array}{c}
\boxed{\boxed{R_{\alpha\beta}-R\frac{g_{\alpha\beta}}{2}}}\\
-\frac{1}{2A}\left(\begin{array}{c}
\left(\begin{array}{c}+A''\cdot a_{,\alpha}a_{,\beta}+A'\cdot a_{,\alpha\beta}-k^{2}\cdot A\cdot L'\cdot l_{,\alpha}\cdot L'\cdot l_{,\beta}\\
+i\cdot k\cdot\left(A'\cdot L'\cdot\left(a_{,\beta}l_{,\alpha}+a_{,\alpha}l_{,\beta}\right)+A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot L'\cdot l_{,\alpha\beta}\right)\end{array}\right)(n-2)\\
+\left(\begin{array}{c}+A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}-k^{2}\cdot A\cdot L'\cdot l_{,a}\cdot L'\cdot l_{,b}\\
+i\cdot k\cdot\left(A'\cdot L'\cdot\left(a_{,b}l_{,a}+a_{,a}l_{,b}\right)+A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\right)\end{array}\right)g_{\alpha\beta}g^{ab}\\
+\left(A'\cdot a_{,a}+i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-\\
\left(A'\cdot a_{,\alpha}+i\cdot k\cdot A\cdot L'\cdot l_{,\alpha}\right)g^{ab}g_{\beta b,a}\\
-\left(A'\cdot a_{,\beta}+i\cdot k\cdot A\cdot L'\cdot l_{,\beta}\right)g^{ab}g_{\alpha b,a}\\
+\left(A'\cdot a_{,d}+i\cdot k\cdot A\cdot L'\cdot l_{,d}\right)g^{cd}\left(\begin{array}{c}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\
+\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)
\end{array}\right)\\
+\frac{1}{4A^{2}}\left(\begin{array}{c}
\left(\begin{array}{c}(A')^{2}a_{,\alpha}a_{,\beta}+i\cdot k\cdot A\cdot L'\cdot A'\left(l_{,\alpha}\cdot a_{,\beta}+a_{,\alpha}l_{,\beta}\right)\\
-k^{2}\cdot(A\cdot L')^{2}\cdot l_{,\alpha}\cdot l_{,\beta}\end{array}\right)(3n-6)\\
+g_{\alpha\beta}\left(\begin{array}{c}(A')^{2}a_{,a}a_{,b}+i\cdot k\cdot A\cdot L'\cdot A'\left(l_{,a}\cdot a_{,b}+a_{,a}l_{,b}\right)\\
-k^{2}\cdot(A\cdot L')^{2}\cdot l_{,a}\cdot l_{,b}\end{array}\right)g^{ab}(4-n)
\end{array}\right)\\
+(n-1)\left(\begin{array}{c}
\frac{g^{ab}}{A}\left(\begin{array}{c}i\cdot k\cdot L'\left(A'\cdot\left(a_{,b}l_{,a}+a_{,a}l_{,b}\right)+A\left(\frac{L''}{L'}l_{,a}l_{,b}+l_{,ab}\right)\right)\\
+A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}-k^{2}\cdot A\cdot L'\cdot l_{,a}\cdot L'\cdot l_{,b}\\
+\frac{\left(A'\cdot a_{,d}+i\cdot k\cdot A\cdot L'\cdot l_{,d}\right)}{2}\left(g^{cd}g_{ab,c}-\frac{ng^{cd}g_{ac,b}}{(n-1)}\right)\end{array}\right)\\
+\frac{g^{ab}\left(\begin{array}{c}(A')^{2}a_{,a}a_{,b}+i\cdot k\cdot A\cdot L'\cdot A'\left(l_{,a}\cdot a_{,b}+a_{,a}l_{,b}\right)\\
-k^{2}\cdot(A\cdot L')^{2}\cdot l_{,a}\cdot l_{,b}\end{array}\right)}{4A^{2}}(n-6)
\end{array}\right)\frac{g_{\alpha\beta}}{2}
\end{array}\right),\qquad(49)
$$

$$
0=\left(\begin{array}{c}
\boxed{R_{\alpha\beta}-R\frac{g_{\alpha\beta}}{2}} \\
-\frac{1}{2A}\left(\begin{array}{c}
\left(A''\cdot a_{,\alpha}a_{,\beta}+A'\cdot a_{,\alpha\beta}\right)(n-2)+\left(A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}\right)g_{\alpha\beta}g^{ab} \\
+A'\left(\begin{array}{c}
a_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-a_{,\alpha}g^{ab}g_{\beta b,a}-a_{,\beta}g^{ab}g_{\alpha b,a} \\
+a_{,d}g^{cd}\left(\begin{array}{c}
g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha} \\
+\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}
\end{array}\right)
\end{array}\right)
\end{array}\right) \\
+(n-1)\left(\begin{array}{c}
\frac{g^{ab}}{A}\left(+A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}+\frac{A'\cdot a_{,d}}{2}\left(g^{cd}g_{ab,c}-\frac{ng^{cd}g_{ac,b}}{(n-1)}\right)\right) \\
+\frac{g^{ab}(A')^{2}a_{,a}a_{,b}}{4A^{2}}(n-6)
\end{array}\right)\frac{g_{\alpha\beta}}{2} \\
+\frac{(A')^{2}}{4A^{2}}\left(a_{,\alpha}a_{,\beta}(3n-6)+g_{\alpha\beta}a_{,a}a_{,b}g^{ab}(4-n)\right) \\
-\frac{i\cdot k}{2A}\left(\begin{array}{c}
\left(A'\cdot L'\cdot\left(a_{,\beta}l_{,\alpha}+a_{,\alpha}l_{,\beta}\right)+A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot L'\cdot l_{,\alpha\beta}\right)(n-2) \\
+\left(A'\cdot L'\cdot\left(a_{,b}l_{,a}+a_{,a}l_{,b}\right)+A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\right)g_{\alpha\beta}g^{ab} \\
+A\cdot L'\cdot\left(\begin{array}{c}
l_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-l_{,\alpha}g^{ab}g_{\beta b,a}-l_{,\beta}g^{ab}g_{\alpha b,a} \\
+l_{,d}g^{cd}\left(\begin{array}{c}
g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha} \\
+\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}
\end{array}\right)
\end{array}\right)
\end{array}\right) \\
+\frac{i\cdot k}{4A}L'A'\left(\left(l_{,\alpha}a_{,\beta}+a_{,\alpha}l_{,\beta}\right)(3n-6)+g_{\alpha\beta}\left(l_{,a}a_{,b}+a_{,a}l_{,b}\right)g^{ab}(4-n)\right) \\
+i\cdot k(n-1)\left(\begin{array}{c}
\frac{g^{ab}}{A}\left(\begin{array}{c}
L'\left(A'\cdot\left(a_{,b}l_{,a}+a_{,a}l_{,b}\right)+A\left(\frac{L''}{L'}l_{,a}l_{,b}+l_{,ab}\right)\right) \\
+\frac{A\cdot L'\cdot l_{,d}}{2}\left(g^{cd}g_{ab,c}-\frac{ng^{cd}g_{ac,b}}{(n-1)}\right)
\end{array}\right) \\
+\frac{g^{ab}L'\cdot A'\left(l_{,a}a_{,b}+a_{,a}l_{,b}\right)}{4A}(n-6)
\end{array}\right)\frac{g_{\alpha\beta}}{2} \\
+\frac{k^{2}}{2}(L')^{2}\left(l_{,\alpha}l_{,\beta}(n-2)+l_{,a}l_{,b}g_{\alpha\beta}g^{ab}\right) \\
-\frac{k^{2}\cdot(L')^{2}}{4}\left(l_{,\alpha}l_{,\beta}(3n-6)+g_{\alpha\beta}l_{,a}l_{,b}g^{ab}(4-n)\right) \\
-k^{2}\cdot(L')^{2}\cdot(n-1)l_{,a}l_{,b}g^{ab}(n-2)\frac{g_{\alpha\beta}}{8}
\end{array}\right). \qquad (50)
$$

Summing up the coefficients for the powers of k gives us:

$$
0=\left(\begin{array}{c}
\boxed{R_{\alpha\beta}-R\frac{g_{\alpha\beta}}{2}}+\frac{(A')^2}{4A^2}a_{,\alpha}a_{,\beta}(3n-6)\\
-\frac{1}{2A}\left(\begin{array}{c}
\left(A''\cdot a_{,\alpha}a_{,\beta}+A'\cdot a_{,\alpha\beta}\right)(n-2)+\left(A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}\right)g_{\alpha\beta}g^{ab}\\
+A'\left(\begin{array}{c}
a_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-a_{,\alpha}g^{ab}g_{\beta b,a}-a_{,\beta}g^{ab}g_{\alpha b,a}\\
+a_{,d}g^{cd}\left(\begin{array}{c}
g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\
+\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}
\end{array}\right)
\end{array}\right)
\end{array}\right)\\
+(n-1)\left(\begin{array}{c}
\left(+A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}+\frac{A'\cdot a_{,d}}{2}\left(g^{cd}g_{ab,c}-\frac{ng^{cd}g_{ac,b}}{(n-1)}\right)\right)\\
+\frac{(A')^2a_{,a}a_{,b}}{4A}\left((n-6)+2\frac{4-n}{n-1}\right)
\end{array}\right)\frac{g^{ab}g_{\alpha\beta}}{2A}\\
-\frac{i\cdot k}{2A}\left(\begin{array}{c}
\left(A'\cdot L'\cdot\left(a_{,\beta}l_{,\alpha}+a_{,\alpha}l_{,\beta}\right)+A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot L'\cdot l_{,\alpha\beta}\right)(n-2)\\
+\left(A'\cdot L'\cdot\left(a_{,b}l_{,a}+a_{,a}l_{,b}\right)+A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\right)g_{\alpha\beta}g^{ab}\\
+A\cdot L'\cdot\left(\begin{array}{c}
l_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-l_{,\alpha}g^{ab}g_{\beta b,a}-l_{,\beta}g^{ab}g_{\alpha b,a}\\
+l_{,d}g^{cd}\left(\begin{array}{c}
g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\
+\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}
\end{array}\right)
\end{array}\right)\\
-\frac{L'A'}{2}\left(\left(l_{,\alpha}a_{,\beta}+a_{,\alpha}l_{,\beta}\right)(3n-6)+g_{\alpha\beta}\left(l_{,a}a_{,b}+a_{,a}l_{,b}\right)g^{ab}(4-n)\right)\\
-(n-1)\left(\begin{array}{c}
L'\left(A'\cdot\left(a_{,b}l_{,a}+a_{,a}l_{,b}\right)+A\left(\frac{L''}{L'}l_{,a}l_{,b}+l_{,ab}\right)\right)\\
+\frac{AL'l_{,d}}{2}\left(g^{cd}g_{ab,c}-\frac{ng^{cd}g_{ac,b}}{(n-1)}\right)\\
+\frac{L'A'\left(l_{,a}a_{,b}+a_{,a}l_{,b}\right)}{4}(n-6)
\end{array}\right)g^{ab}g_{\alpha\beta}
\end{array}\right)\\
-\frac{k^2}{8}(n-2)(L')^2\left(l_{,a}l_{,b}g_{\alpha\beta}g^{ab}(n-3)+2l_{,\alpha}l_{,\beta}\right)
\end{array}\right). \qquad (51)
$$

Consequently, for big k, we would obtain the following Eikonal equation:

$$
0=l_{,a}l_{,b}g_{\alpha\beta}g^{ab}(n-3)+2l_{,\alpha}l_{,\beta}. \qquad (52)
$$

and the quantum gravity equivalent for the classical equation (9) as follows:

$$0=\left(\begin{array}{c}\left(A'\cdot L'\cdot\left(a_{,\beta}l_{,\alpha}+a_{,\alpha}l_{,\beta}\right)+A\cdot L''\cdot l_{,\alpha}l_{,\beta}+A\cdot L'\cdot l_{,\alpha\beta}\right)(n-2)\\ +\left(A'\cdot L'\cdot\left(a_{,b}l_{,a}+a_{,a}l_{,b}\right)+A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\right)g_{\alpha\beta}g^{ab}\\ +A\cdot L'\cdot\left(\begin{array}{c}l_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-l_{,\alpha}g^{ab}g_{\beta b,a}-l_{,\beta}g^{ab}g_{\alpha b,a}\\ +l_{,d}g^{cd}\left(\begin{array}{c}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)\end{array}\right)\\ -\frac{L'A'}{2}\left(\left(l_{,\alpha}a_{,\beta}+a_{,\alpha}l_{,\beta}\right)(3n-6)+g_{\alpha\beta}\left(l_{,a}a_{,b}+a_{,a}l_{,b}\right)g^{ab}(4-n)\right)\\ -(n-1)\left(\begin{array}{c}L'\left(A'\cdot\left(a_{,b}l_{,a}+a_{,a}l_{,b}\right)+A\left(\frac{L''}{L'}l_{,a}l_{,b}+l_{,ab}\right)\right)\\ +\frac{AL'l_{,d}}{2}\left(g^{cd}g_{ab,c}-\frac{ng^{cd}g_{ac,b}}{(n-1)}\right)+\frac{L'A'\left(l_{,a}a_{,b}+a_{,a}l_{,b}\right)}{4}(n-6)\end{array}\right)g^{ab}g_{\alpha\beta}\end{array}\right).\quad(53)$$

Interestingly, the gravity terms containing the Ricci curvatures (the vacuum Einstein field equations; see boxed terms):

$$\left(\begin{array}{c}\boxed{R_{\alpha\beta}-R\frac{g_{\alpha\beta}}{2}}+\frac{(A')^{2}}{4A^{2}}a_{,\alpha}a_{,\beta}(3n-6)\\ -\frac{1}{2A}\left(\begin{array}{c}\left(A''\cdot a_{,\alpha}a_{,\beta}+A'\cdot a_{,\alpha\beta}\right)(n-2)+\left(A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}\right)g_{\alpha\beta}g^{ab}\\ +A'\left(\begin{array}{c}a_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-a_{,\alpha}g^{ab}g_{\beta b,a}-a_{,\beta}g^{ab}g_{\alpha b,a}\\ +a_{,d}g^{cd}\left(\begin{array}{c}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)\end{array}\right)\end{array}\right)\\ +(n-1)\left(\begin{array}{c}\left(+A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}+\frac{A'\cdot a_{,d}}{2}\left(g^{cd}g_{ab,c}-\frac{ng^{cd}g_{ac,b}}{(n-1)}\right)\right)\\ +\frac{(A')^{2}a_{,a}a_{,b}}{4A}\left((n-6)+2\frac{4-n}{n-1}\right)\end{array}\right)\frac{g^{ab}g_{\alpha\beta}}{2A}\end{array}\right).\quad(54)$$

are becoming recessive.

It should be pointed out that, as before, with a suitable choice for L, one could get rid of the term $l_{,\alpha}l_{,\beta}$ in the quantum gravity Eikonal equation (see next sub-section). This however, will make the function L k-dependent and compromises our Eikonal derivation from above.

3.1.2.1.1 The Other "Eikonal"

Taking (21), we have seen in [8] that we have a variety of options to obtain various n-dependencies for our wrapping function F[f]. Leaving F open and assuming that it does not depend on n, results in:

$$0=\left[\begin{array}{c}\left(R_{\alpha\beta}-\frac{1}{2F}\left(\begin{array}{c}-2F_{,\alpha\beta}+F_{,ab}g_{\alpha\beta}g^{ab}-F_{,\beta}g^{ab}g_{\alpha b,a}\\+F_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-F_{,\alpha}g^{ab}g_{\beta b,a}\\+F_{,d}g^{cd}\left(g_{\alpha c,\beta}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\right)\end{array}\right)\right.\\ \left.+\frac{1}{2F^{2}}\left(-3F_{,\alpha}\cdot F_{,\beta}+2g_{\alpha\beta}F_{,c}F_{,d}g^{cd}\right)\right)\\ -\left(R+\frac{1}{2F}\left(2g^{ab}F_{,ab}+F_{,d}g^{cd}g^{ab}g_{ab,c}\right)-3\frac{g^{ab}F_{,a}\cdot F_{,b}}{2F^{2}}\right)\cdot\frac{g_{\alpha\beta}}{2}\\ +n\left[\begin{array}{c}\left(\begin{array}{c}-\frac{1}{2F}\left(F_{,\alpha\beta}-\frac{F_{,d}}{2}g^{cd}\left(g_{\alpha c,\beta}+g_{\beta c,\alpha}-g_{\alpha\beta,c}\right)\right)\\+\frac{1}{4F^{2}}\left(3F_{,\alpha}\cdot F_{,\beta}-g_{\alpha\beta}F_{,c}F_{,d}g^{cd}\right)\end{array}\right)\\+\left(\begin{array}{c}\frac{1}{2F}\left(2g^{ab}F_{,ab}+F_{,d}g^{cd}g^{ab}g_{ab,c}-F_{,d}g^{cd}g^{ab}g_{ac,b}\right)\\-7\frac{g^{ab}F_{,a}\cdot F_{,b}}{4F^{2}}\end{array}\right)\cdot\frac{g_{\alpha\beta}}{2}\end{array}\right]+n^{2}\frac{g^{ab}F_{,a}\cdot F_{,b}}{4F^{2}}\cdot g_{\alpha\beta}\end{array}\right]. \tag{55}$$

This quadratic polynomial in n shows the following dominant term as limit of n→∞:

$$0=\frac{g^{ab}F_{,a}\cdot F_{,b}}{8F^{2}}\cdot g_{\alpha\beta}\,. \tag{56}$$

We recognize the scalar Eikonal equation reading:

$$0=g^{ab}F_{,a}\cdot F_{,b}\,. \tag{57}$$

This equation is by far less restricted than (52) and it also much better mirrors the classical one (8). De facto, when remembering the b²-term can be easily constructed via a suitable additional dimension (see section "The Classical Eikonal Equation Derived From a Quantum Gravity Origin"), it actually is the classical Eikonal equation.

While as before the Ricci-curvature terms:

$$\left(\begin{array}{c}\left(R_{\alpha\beta}-\frac{1}{2F}\left(\begin{array}{c}-2F_{,\alpha\beta}+F_{,ab}g_{\alpha\beta}g^{ab}-F_{,\beta}g^{ab}g_{\alpha b,a}\\+F_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-F_{,\alpha}g^{ab}g_{\beta b,a}\\+F_{,d}g^{cd}\left(g_{\alpha c,\beta}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\right)\end{array}\right)\right.\\ \left.+\frac{1}{2F^{2}}\left(-3F_{,\alpha}\cdot F_{,\beta}+2g_{\alpha\beta}F_{,c}F_{,d}g^{cd}\right)\right)\\ -\left(R+\frac{1}{2F}\left(2g^{ab}F_{,ab}+F_{,d}g^{cd}g^{ab}g_{ab,c}\right)-3\frac{g^{ab}F_{,a}\cdot F_{,b}}{2F^{2}}\right)\cdot\frac{g_{\alpha\beta}}{2}\end{array}\right). \tag{58}$$

become recessive, we find the equivalent to the classical Eikonal-"side-kick" (9) via:

$$0=\left(\begin{array}{c} -\frac{1}{2F}\left(F_{,\alpha\beta}-\frac{F_{,d}}{2}g^{cd}\left(g_{\alpha c,\beta}+g_{\beta c,\alpha}-g_{\alpha\beta,c}\right)\right) \\ +\frac{1}{4F^2}\left(3F_{,\alpha}\cdot F_{,\beta}-g_{\alpha\beta}F_{,c}F_{,d}g^{cd}\right) \\ +\left(\frac{1}{2F}\left(2g^{ab}F_{,ab}+F_{,d}g^{cd}g^{ab}g_{ab,c}-F_{,d}g^{cd}g^{ab}g_{ac,b}\right)-7\frac{g^{ab}F_{,a}\cdot F_{,b}}{4F^2}\right)\cdot\frac{g_{\alpha\beta}}{2} \end{array}\right). \tag{59}$$

3.1.3 Special Consideration of the Non-Linear Terms

Reordering (45) with respect to the linear and non-linear terms in a and l, gives us:

$$0=\left(\begin{array}{c} \left(\frac{3(A')^2}{2A}-A''\right)\cdot a_{,\alpha}a_{,\beta}-A'\cdot a_{,\alpha\beta} \\ +g_{\alpha\beta}g^{ab}\left(\left(\frac{(A')^2}{4A}(n-7)+A''\right)\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}\right) \\ +i\cdot k\cdot L'\cdot\left(\begin{array}{c} \frac{A'\cdot\left(a_{,\alpha}l_{,\beta}+l_{,\alpha}a_{,\beta}\right)}{2}-A\cdot l_{,\alpha\beta} \\ +g_{\alpha\beta}g^{ab}\left(A'\cdot\left(a_{,a}l_{,b}+a_{,b}l_{,a}\right)\left(1+\frac{n-7}{4}\right)+A\cdot l_{,ab}\right) \end{array}\right) \\ +g_{\alpha\beta}g^{ab}A\cdot l_{,a}\cdot l_{,b}\left(i\cdot k\cdot L''+\frac{k^2(L')^2}{4}\cdot(n-3)\right) \\ +l_{,\alpha}\cdot l_{,\beta}A\cdot\left(i\cdot k\cdot L''-\frac{k^2(L')^2}{2}\right) \end{array}\right). \tag{60}$$

With the setting for F of the kind:

$$\begin{array}{c} A\left[a[\ldots]\right]=\frac{C_F}{\left(a[\ldots]+C_a\right)^2};\quad L\left[l[\ldots]\right]=C_L+\ln\left(k\cdot l[\ldots]+C_l\right)^{\frac{2i}{k}} \\ \Rightarrow F=C_F\left(\frac{\left(k\cdot l[\ldots]+C_l\right)^{\frac{i}{k}}}{a[\ldots]+C_a}\right)^2 \end{array}, \tag{61}$$

(60) simplifies to:

$$0=\left(\begin{array}{c} g_{\alpha\beta}g^{ab}\left(\left(\frac{(A')^2}{4A}(n-7)+A''\right)\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}\right) \\ +i\cdot k\cdot L'\cdot\left(\begin{array}{c} \frac{A'\cdot\left(a_{,\alpha}l_{,\beta}+l_{,\alpha}a_{,\beta}\right)}{2}-A\cdot l_{,\alpha\beta} \\ +g_{\alpha\beta}g^{ab}\left(A'\cdot\left(a_{,a}l_{,b}+a_{,b}l_{,a}\right)\left(1+\frac{n-7}{4}\right)+A\cdot l_{,ab}\right) \end{array}\right) \\ +g_{\alpha\beta}g^{ab}A\cdot l_{,a}\cdot l_{,b}\left(i\cdot k\cdot L''+\frac{k^2(L')^2}{4}\cdot(n-3)\right) \end{array}\right). \tag{62}$$

With this F, not being dependent on the dimensionality n, we find an interesting situation when considering space and space-times with huge numbers of dimensions. In these cases the equation (62) evolves to:

$$0=\frac{g_{\alpha\beta}g^{ab}}{4}\left(\frac{(A')^{2}}{A}\cdot a_{,a}a_{,b}+i\cdot k\cdot L'\cdot A'\cdot\left(a_{,a}l_{,b}+a_{,b}l_{,a}\right)+A\cdot l_{,a}\cdot l_{,b}k^{2}(L')^{2}\right). \tag{63}$$

It should be pointed out that we obtain the same result with any other F not depending on n.

3.2 The Classical Eikonal Equation Derived From a Quantum Gravity Origin in the Classical Way

Starting with (15) and applying the setting (43) for the function f as follows:

$$f=A\left[a\left[x_{i}\right]\right]\cdot e^{i\cdot k\cdot L\left[l\left[x_{i}\right]\right]}, \tag{64}$$

we obtain from (30):

$$\begin{array}{l}
0=R-\dfrac{2}{(n-2)\left(f+C_{f}\right)}\left(\begin{array}{c}
(n-1)\left(\begin{array}{c}
2g^{ab}\left(\begin{array}{c}
\left(A'\cdot a_{,a}+i\cdot k\cdot A\cdot L'\cdot l_{,a}\right)\cdot i\cdot k\cdot L'\cdot l_{,b}\\
+\left(\begin{array}{c}
A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}\\
+A'\cdot a_{,b}\cdot i\cdot k\cdot L'\cdot l_{,a}\\
+i\cdot k\cdot A\cdot L''\cdot l_{,a}l_{,b}+i\cdot k\cdot A\cdot L'\cdot l_{,ab}
\end{array}\right)
\end{array}\right)\\
+\left(A'\cdot a_{,d}+A\cdot i\cdot k\cdot L'\cdot l_{,d}\right)g^{cd}g^{ab}g_{ab,c}
\end{array}\right)\\
-n\left(A'\cdot a_{,d}+A\cdot i\cdot k\cdot L'\cdot l_{,d}\right)g^{cd}g^{ab}g_{ac,b}
\end{array}\right)\cdot e^{L[l]}\\
\xrightarrow{f+C_{f}\to f}\\
0=R-\dfrac{2}{(n-2)A}\left(\begin{array}{c}
(n-1)\left(\begin{array}{c}
2g^{ab}\left(\begin{array}{c}
\left(A'\cdot a_{,a}\cdot i\cdot k\cdot L'\cdot l_{,b}-k^{2}\cdot A\cdot(L')^{2}\cdot l_{,b}\cdot l_{,a}\right)\\
+\left(\begin{array}{c}
A''\cdot a_{,a}a_{,b}+A'\cdot a_{,ab}\\
+i\cdot k\cdot\left(A'\cdot a_{,b}\cdot L'\cdot l_{,a}+A\cdot L''\cdot l_{,a}l_{,b}+A\cdot L'\cdot l_{,ab}\right)
\end{array}\right)
\end{array}\right)\\
+\left(A'\cdot a_{,d}+A\cdot i\cdot k\cdot L'\cdot l_{,d}\right)g^{cd}g^{ab}g_{ab,c}
\end{array}\right)\\
-n\left(A'\cdot a_{,d}+A\cdot i\cdot k\cdot L'\cdot l_{,d}\right)g^{cd}g^{ab}g_{ac,b}
\end{array}\right)
\end{array}. \tag{65}$$

Assuming huge k, the dominant term would be:

$$\begin{aligned}
0&=-\frac{2}{(n-2)A}\left((n-1)\left(2g^{ab}k^{2}\cdot A\cdot(L')^{2}\cdot l_{,b}\cdot l_{,a}\right)\right)\\
&=-\frac{4(n-1)}{(n-2)}g^{ab}k^{2}\cdot(L')^{2}\cdot l_{,b}\cdot l_{,a}
\end{aligned}. \tag{66}$$

This is the quantum gravity derived Eikonal equation, but for those who miss the constant term in comparison to the classical Eikonal equation (8), we refer to [8], where it is shown how such a constant arises from just a another dimension.

4 Consequences

The classical Eikonal equation can either be obtained from the Einstein-Hilbert action in the classical way, which is to say with an approach of the type (28) plus the assumption of huge wave numbers k, from the full quantum gravity field equations (12) by a volume restricted variation (weak gravity condition) (13) or via the assumption of huge numbers of dimensions n. While in the latter case no additional conditions are of need, we have a variety of such in the first case. This is bit of a surprise, because classically the Einkonal equation is derived from a wave equation only via the assumption of big k. With the Quantum Gravity starting point, however, when only taking the condition of big k (without the weak gravity condition), we'd end up with a non-classical Eikonal equation in the form of (52) (new and fully quantum gravity derived) in comparison with the much simpler form (66) (perfectly agreeing with the classical Eikonal form) for large n.

It is very interesting that we obtain a classical equation of particles (particles of light) from the quantum gravity field equations via a limiting procedure concerning the dimensionality of the system rather than the classical restriction of the wave function approach (restriction to big wave numbers). After all, there is no obvious connection between big wave numbers and high system dimensionality. Our biological/evolutionary preconditioning forces us to try and see connections in cases where the results are equal and so, consequently, we also tend to intent to see such a connection here and ask "What could it possibly be that makes the two completely different limiting procedures to produce the same result?". There seems to be no direct association between high dimensionality of a system and big wave numbers for solutions within this very system. And in fact, there is no such connection (except that it gives the same limiting result from a quantum gravity standpoint)… or is there? Well, in systems of high dimensionality the likelihood for the observation of small wavelengths is probably higher, because a greater number of dimensions means a greater number of degrees of freedom and those can cover oscillations the observer sees as concentrated or compactified on smaller ensembles of dimensions (manifolds), which apparently is equal to smaller wavelengths.

Still, the rather different Eikonal equations for big k (52) in comparison with the much simpler form (66) for big n, both resulting from the same Quantum Gravity starting point, shows us that there can be no perfect relationship between the two procedures. The equivalence in the outcome in the case of weak gravity is a triviality, because by restricting the variation to just the volume of the metric we have reduced the Quantum Gravity to just its quantum part and thus, a wave equation, which, de facto, is the classical starting point for the derivation of the Eikonal equation. Consequently, of course there can be no other outcome there when setting in a general wave of type (1) and approximating the equation with respect to big k. That the same result is obtained from the full Quantum Gravity equations via big n is a structural coincidence for n roughly playing the role in the full equations, k is playing in the restricted (weak gravity) equations. So, shouldn't one conclude now that n can be seen as some kind of wave number or k as some kind of dimensionality?

Yes and No!

Yes, under the right circumstances (no internal structural variation of the metric but only volume variation), the wave number k is as global as the parameter n, because it affects always the whole volume or scaling function f, which entangles all dimensions and not just an internal manifold.

No, when taking the full Quantum Gravity equations as starting point the outcome for the limiting procedures for k (52) and n (66) are significantly different.

We can prove this by investigating sub-systems with chain-ordered or nested scaling functions $F_i[f_i]$ in the same way as we did it here for the global scaling or volume function. The interested reader can find the necessary math in [6, 7, 10] in connection with our quantum gravity consideration of Hugh Everett's multiverse theory.

5 Applications

As there are infinitely many systems with huge numbers of degrees of freedom, we can formulate an equation for the expectation value E[…] for possible applications E[Applications] as follows [11, 12]:

$$E\left[\text{Applications}\right]=\frac{\int\limits_{\text{Universe}} d^n x\sqrt{|g|}\cdot F^{\frac{n}{4}*}\cdot \text{Applications}\cdot F^{\frac{n}{4}}}{\int\limits_{\text{Universe}} d^n x\sqrt{|g|}\cdot F^{\frac{n}{4}*}\cdot F^{\frac{n}{4}}}=\infty\,! \qquad (67)$$

6 Conclusions

We have demonstrated in this short paper how the classical Eikonal equation can be derived from a quantum gravity theory in a completely non-classical way. Thereby one only needs to assume a huge number of degrees of freedom or dimensions residing within the system of interest. The classical way, where a general wave approach is used and big wave numbers are assumed, does not produce the classical but a slightly more complicated Eikonal equation. In this case, only certain additional boundary conditions, like the assumption of a variation for the Einstein-Hilber action being restricted to only the metric scaling or volume of the system, will bring the usual, classical result.

7 References

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