The Harmonic Sphere

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Abstract

Starting with the solutions to a classical harmonic potential in radial direction, fully metrically derived from an extended Einstein-Hilbert action [A1, A2], thereby using our Quantum Gravity approach [A3 – A8], we demonstrate that the resulting quantum fields are those of half-spin objects. In addition to this half-spin character, we also find that the resulting "particles" have a rather well-defined finite or wall structure, which is in clear contrast to – e.g. – the well-distributed orbital solutions to the hydrogen atom. This clear boundary demarcation might be seen as a typical characteristic of particles, which are in this case, due to the half-spin conditions, fermions.

Abstract References

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Metric Construction of the Spherical Harmonic Oscillator

Using the method laid out in previous publications [1 - 9] and partially repeated for convenience the appendix of this paper, we start with the following metric tensor in 8 dimensions:

$$g^{n}_{\alpha\beta}=\begin{pmatrix} -c^2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & r^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & r^2\cdot\sin^2\varphi_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{n-1n-1} \end{pmatrix}$$

$$\times F\left[f\left[t,r,\vartheta,\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5\right]\right];\quad n=8;\quad F[f]=f^{\frac{4}{n-2}}=f^{\frac{2}{3}}$$

$$f\left[t,r,\vartheta,\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5\right]=f_t[t]\cdot f_r[r]\cdot f_\vartheta[\vartheta]\cdot\prod_{i=1}^{5}f_{\varphi_i}[\varphi_i]; \tag{1}$$

$$g_{44}=\left(g_{\varphi_2}[\varphi_2]'\right)^2\cdot k[r]^{i\cdot s};\quad g_{55}=\left(g_{\varphi_3}[\varphi_3]'\right)^2\cdot k[r]^{s};$$

$$g_{66}=\left(g_{\varphi_4}[\varphi_4]'\right)^2\cdot k[r]^{-i\cdot s};\quad g_{77}=\left(g_{\varphi_5}[\varphi_5]'\right)^2\cdot k[r]^{-s}$$

Evaluation of the corresponding Ricci scalar and following the procedure in the appendix, thereby using the separation approach and the partial solutions as given in the appendix in equations (17) to (19), we obtain the following scalarized Quantum Field equation for the radius coordinate r:

$$R^*=0=\begin{pmatrix} \frac{L\cdot(L+1)}{r^2}+k[r]^{i\cdot s}\cdot A_2^2+k[r]^{s}\cdot A_3^2 \\ +k[r]^{-i\cdot s}\cdot A_4^2+k[r]^{-s}\cdot A_5^2-\overbrace{\Delta_{3D-sphere}[r]}^{2\frac{\partial}{r\cdot\partial r}+\frac{\partial^2}{\partial r^2}}-E \end{pmatrix}\Psi$$

$$\Psi=\Psi\left[r,\vartheta=\varphi_0,\varphi=\varphi_1\right]=f_r[r]\cdot f_\vartheta\left[\vartheta=\varphi_0\right]\cdot f_\varphi\left[\varphi=\varphi_1\right]\quad . \tag{2}$$

$$\Rightarrow$$

$$0=f_\vartheta[\vartheta]\cdot f[\varphi]\cdot\begin{pmatrix} \frac{L\cdot(L+1)}{r^2}+k[r]^{i\cdot s}\cdot A_2^2+k[r]^{s}\cdot A_3^2+k[r]^{-i\cdot s}\cdot A_4^2 \\ +k[r]^{-s}\cdot A_5^2-\frac{2\partial}{r\cdot\partial r}-\frac{\partial^2}{\partial r^2}-E \end{pmatrix}f_r[r]$$

We see that for the choice of k[r] as:

$$k[r]=V^2\frac{r^2}{A_3^2}, \tag{3}$$

s=1 and $f_{\varphi_i}\left[\varphi_i\right]=\text{const}$ for i=2,4,5, we obtain:

$$0=f_\vartheta[\vartheta]\cdot f[\varphi]\cdot\left(\frac{L\cdot(L+1)}{r^2}+V^2\cdot r^2-\frac{2\partial}{r\cdot\partial r}-\frac{\partial^2}{\partial r^2}+E\right)f_r[r]$$

$$\Rightarrow\quad . \tag{4}$$

$$\left(-\frac{L\cdot(L+1)}{r^2}-V^2\cdot r^2+2\frac{\partial}{r\cdot\partial r}+\frac{\partial^2}{\partial r^2}\right)f_r[r]=E\cdot f_r[r]$$

We obtain the following solution for f_r:

$$f_r[r] = \frac{2^{\frac{1}{2}\left(\frac{1}{2}-L\right)} e^{-\frac{1}{2}\cdot V\cdot r^2} \cdot r^{-L}}{r}\left(C_U \cdot U\left[-q, \frac{1}{2}-L, V\cdot r^2\right] + C_L \cdot L_q^{-L-\frac{1}{2}}\left[V\cdot r^2\right]\right). \qquad (5)$$
$$q = -\frac{E+V-2\cdot V\cdot L}{4\cdot V}$$

We recognize the hypergeometric function U[a,b,z] and the Laguerre polynomials $L_q^{-L-\frac{1}{2}}$ in the solutions above. These we already know from the Schrödinger hydrogen solution (e.g., [10]). The only reasonable solutions (singularity free) can be found for the case of half spins $L=\ell/2$ and then (5) simplifies to:

$$f_r[r] = \frac{2^{\frac{1}{4}(1-\ell)} e^{-\frac{1}{2}\cdot V\cdot r^2} \cdot r^{-\frac{\ell}{2}}}{r}\left(C_U \cdot U\left[-q, \frac{1-\ell}{2}, V\cdot r^2\right] + C_L \cdot L_q^{-\frac{1+\ell}{2}}\left[V\cdot r^2\right]\right). \qquad (6)$$
$$q = -\frac{E+V-V\cdot\ell}{4\cdot V} = \frac{1}{4}\cdot\left(\ell-1-\frac{E}{V}\right)$$

Figure 1 shows the corresponding f_r-distribution for six settings ℓ=1/2…11/2 and the corresponding q=1…6 with C_U=0 and C_L=1.

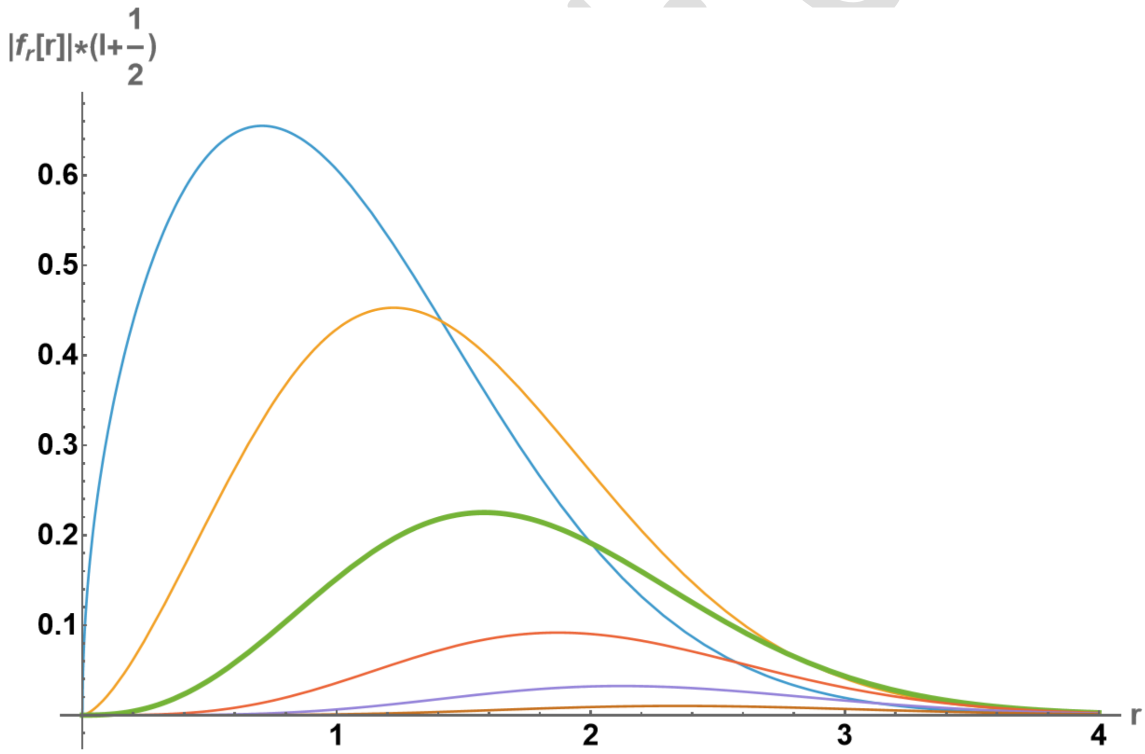

Fig. 1: f_r-distribution from (6) for six settings ℓ=1/2…11/2 and the corresponding q=1…6 with C_U=0 and C_L=1.

We see that – in contrast to the Schrödinger hydrogen solutions (8) – our harmonic radial potential solution shows a very rigid behavior regarding the demarcation of the object at a certain distance from the center. All the usual "outside oscillation" as we know it from the 1/r-potential has disappeared. One might see these solutions therefore as particles… especially as they sport half spins, an aspect, which we will consider in the next section.

About Spin 1/2, 3/2, 5/2 and so on

In the following we want to investigate potential half-spin solutions for the associated Legendre polynomials, which we need in various forms within this paper (see section above).

Here, as an example, we intend to consider the simple form:

$$Y_l^m[v] = C_{Pv} \cdot P_l^m[\cos[v]] + C_{Qv} \cdot Q_l^m[\cos[v]], \qquad (7)$$

Observing the solution Y[v] more closely, we find that there exist singularity-free spin l=n/2-solutuions for m=-n/2 with n=1,3,5,7… in the case of $C_{Pv}\neq 0$, $C_{Qv}=0$ and for m=+n/2 in the case of $C_{Pv}=0$, $C_{Qv}\neq 0$. Figures 2 and 3 illustrate the corresponding distribution within the domain of definition for the angle v from 0 to π.

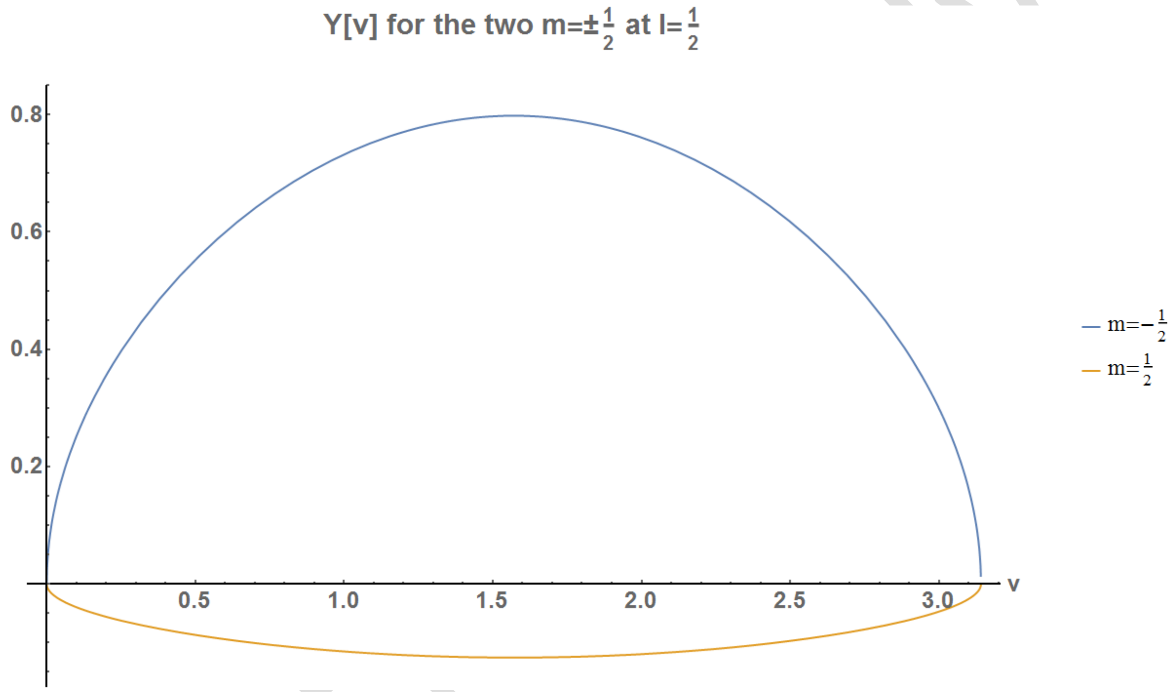

Fig. 2: Spin l=½ situation with the two possible spin states m=±1/2 according to our angular Quantum Gravity solution (7). For better illustration and comparability, we divided the "Q-Legendres" by 10.

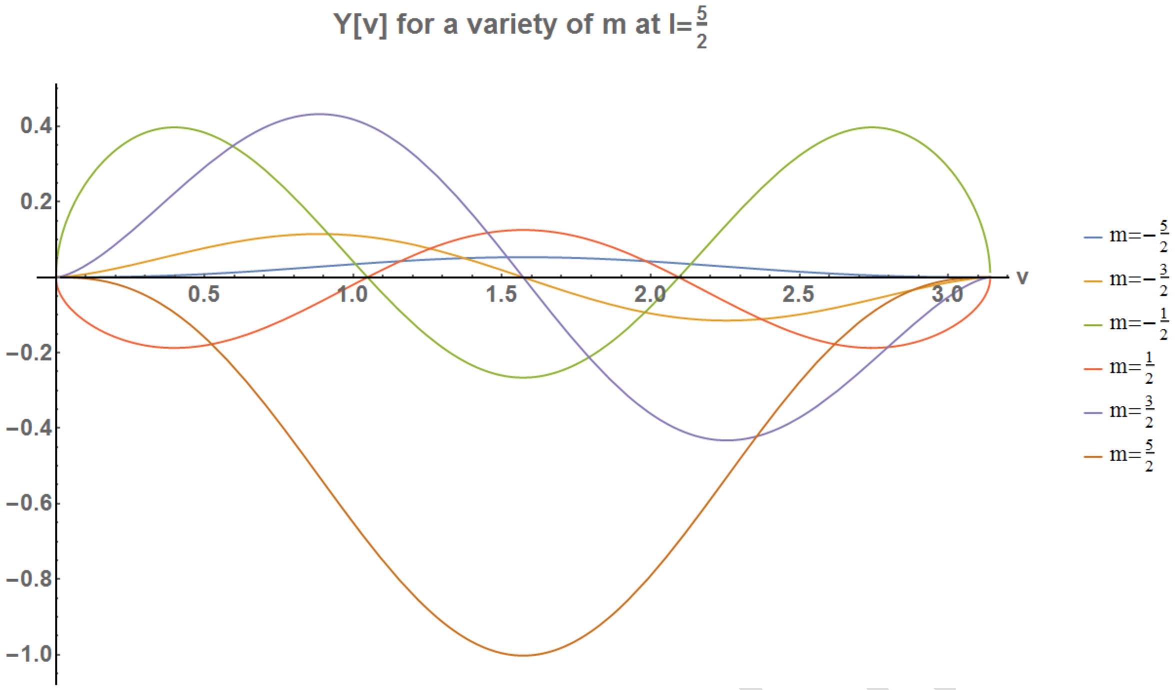

Fig. 3: Spin l=5/2 situation with the five possible spin states m=±{1/2,3/2,5/2} according to our angular Quantum Gravity solution (7). Regarding the evaluation, see text. For better illustration we divided the "Q-Legendres" by 10.

The general solution to the partial differential equation resulting from the Schrödinger hydrogen problem [10, 11, 12] can be given as follows:

$$\begin{aligned} f_{n,l,m}\left[t,r,\vartheta,\varphi\right] &= g\left[t\right]\cdot\Psi_{n,l,m}\left[r,\vartheta,\varphi\right] = g\left[t\right]\cdot e^{i\cdot m\cdot\varphi}\cdot P_l^m\left[\cos\vartheta\right]\cdot R_{n,l}\left[r\right] \\ &= \left(C_1\cdot\cos\left[c\cdot C_{tt}\cdot t\right]+C_2\cdot\sin\left[c\cdot C_{tt}\cdot t\right]\right)\cdot N\cdot e^{-\rho/2}\cdot\rho^l\cdot L_{n-l-1}^{2l+1}\left[\rho\right]\cdot Y_l^m\left[\vartheta,\varphi\right]. \\ \rho &= \frac{2\cdot r}{n\cdot a_0};\quad N=\sqrt{\left(\frac{2}{n\cdot a_0}\right)^3\frac{\left(n-l-1\right)!}{2\cdot n\cdot\left(n+l\right)!}} \end{aligned} \tag{8}$$

It totally suffices here to consider the classical problems in order to also discuss the angular aspects of more complicated metrics, like:

$$\begin{aligned} G_{\alpha\beta}^{8} &= \begin{pmatrix} -c^2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & r^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & r^2\cdot\sin^2\varphi_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{77} \end{pmatrix}, \\ &\times F\left[f\left[t,r,\vartheta,\varphi,\tau,\rho,\theta,\phi\right]\right];\quad F[f]=f^{\frac{2}{3}} \\ f\left[t,r,\vartheta,\varphi,\tau,\rho,\theta,\phi\right] &= f_t\left[t\right]\cdot f_r\left[r\right]\cdot f_\vartheta\left[\vartheta\right]\cdot f_\varphi\left[\varphi\right]\cdot f_\tau\left[\tau\right]\cdot f_\rho\left[\rho\right]\cdot f_\theta\left[\theta\right]\cdot f_\phi\left[\phi\right]; \\ g_{44} &= c_4^2\cdot r^{-2};\quad g_{55}=r^2;\quad g_{66}=r^2\cdot\rho^2;\quad g_{77}=r^2\cdot\rho^2\cdot\sin\left[\theta\right]^2; \end{aligned} \tag{9}$$

for instance, as the results for these coordinates in both cases are the same functional dependencies.

For nostalgic reasons we used the symbol “n” (here not the dimension) as the so-called main quantum number. All quantum numbers n, l and the parameter a_0 depend on the constants E, A_i in connection with our solutions to metric (9) above and have to satisfy certain quantum conditions in order to result in singularity-free solutions for f[…].

Regarding the conditions for the quantum numbers n, l and m, we not only have the usual:

$$\{n,l,m\}\in\mathbb{Z};\quad n\geq 0;\quad l<n;\quad -l\leq m\leq +l, \tag{10}$$

but also found the suitable solutions for the half-spin forms as discussed above and derived in the figures 2 and 3. The corresponding main quantum numbers for half spin l-numbers with l=1/2, 3/2,… are simply (just as before with the integers) n=l+1=3/2,5/2,7/2,….

It should explicitly be noted, however, that the usual spherical harmonics are inapplicable in cases of half spin. For {n,l,m}={1/2,3/2,5/2,7/2,…} the wave function (8) has to be adapted as follows:

$$\begin{aligned} f_{n,l,m}[t,r,\vartheta,\varphi] &= e^{\pm i\cdot c\cdot C_t\cdot t}\cdot N\cdot e^{-\rho/2}\cdot\rho^{l}\cdot L_{n-l-1}^{2l+1}[\rho]\cdot Z_m[\varphi]\cdot\begin{Bmatrix} P_l^{m<0}[\cos\vartheta]\\ Q_l^{m>0}[\cos\vartheta]\end{Bmatrix}\\ &= e^{\pm i\cdot c\cdot C_t\cdot t}\cdot N\cdot e^{-\rho/2}\cdot\rho^{l}\cdot L_{n-l-1}^{2l+1}[\rho]\cdot\begin{Bmatrix}\cos[m\cdot\varphi]\\ \sin[m\cdot\varphi]\end{Bmatrix}\cdot\begin{Bmatrix} P_l^{m<0}[\cos\vartheta]\\ Q_l^{m>0}[\cos\vartheta]\end{Bmatrix}\end{aligned}. \tag{11}$$

Thereby it was elaborated elsewhere [13] that in fact the sin- and the cos-functions seem to make the Pauli exclusion [14] and not the “+” and “-“ of the m. However, in order to have the usual Fermionic statistic we can simply define as follows:

$$f_{n,l,m}[t,r,\vartheta,\varphi] = e^{\pm i\cdot c\cdot C_t\cdot t}\cdot N\cdot e^{-\rho/2}\cdot\rho^{l}\cdot L_{n-l-1}^{2l+1}[\rho]\cdot\begin{Bmatrix}\sin[m\cdot\varphi]_{m<0}\\ \cos[m\cdot\varphi]_{m>0}\end{Bmatrix}\cdot\begin{Bmatrix} P_l^{m<0}[\cos\vartheta]\\ Q_l^{m>0}[\cos\vartheta]\end{Bmatrix}. \tag{12}$$

As derived in [13], the resolution of the degeneration with respect to half spin requires a break of the symmetry, which we achieved by introducing elliptical geometry instead of the spherical one in [13].

Thus, we have metrically derived a fairly general “hydrogen atom”. In addition to the Schrödinger structure, our form also sports a time-dependent factor clearly showing the options for matter and anti-matter via the $\pm$-sign in the g[t]-function. We also found the half spin states and were able to resolve the spin-degeneration via a simple symmetry break by switching from spherical to elliptical coordinates (see [13] regarding the evaluation). A few illustrations of the half-spin solutions are given in [15] in the section “Illustration of Spin-1/2-Solutions”.

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Appendix: Obtaining Purely Metric Radial Potentials

It was shown in [1, 2, 3] that in connection with condition:

$$\delta G^{\alpha\beta} = G^{\alpha\beta}\cdot\delta_0 + \overbrace{G^{ab}\delta^{\alpha\beta}_{ab}}^{\text{Gravity}} \xrightarrow{\forall\,\delta^{\alpha\beta}_{ab}\ll\delta_0} = \frac{g^{\alpha\beta}}{F}\cdot\delta_0\,, \qquad (13)$$

and the subsequent scalarized Quantum Einstein Field Equations:

$$0 = R - \frac{n-1}{2F}\left(\begin{array}{c}\left(\overbrace{2g^{ab}F_{,ab} + F_{,d}g^{cd}g^{ab}g_{ab,c}}^{=2\Delta F - 2F_{,d}g^{cd}{}_{,c}}\right)\\ -nF_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right) - \frac{g^{ab}F_{,a}\cdot F_{,b}}{4F^2}(n-6)\cdot \qquad (14)$$

a potential can be constructed via four additional dimensions. We demonstrated that we can obtain any desired shape of potential in r, via a set of additional dimensions (coordinates) with always four such coordinates for each exponent in r. So, being interested in a potential of the type: const, 1/r, $1/r^2$, for instance, our additional metric components should look as follows:

$$\begin{gathered}
g^{12}_{\alpha\beta} = \begin{pmatrix} -c^2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & r^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & r^2\cdot\sin^2\varphi_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{n-1n-1} \end{pmatrix} \\
\times F\left[f\left[t,r,\vartheta,\varphi_1,\varphi_2,\varphi_3,\varphi_4,\ldots,\varphi_9\right]\right];\quad n=12;\quad F[f]=f^{\frac{4}{12-2}}=f^{\frac{2}{5}} \\
f\left[t,r,\vartheta,\varphi_1,\varphi_2,\varphi_3,\varphi_4,\ldots,\varphi_9\right]=f_t[t]\cdot f_r[r]\cdot f_\vartheta[\vartheta]\cdot\prod_{i=1}^{9} f_{\varphi_i}[\varphi_i]; \\
s=1\quad\begin{cases} g_{44}=\left(g_{\varphi_2}[\varphi_2]'\right)^2\cdot r^{i\cdot s};\quad g_{55}=\left(g_{\varphi_3}[\varphi_3]'\right)^2\cdot r^{s}; \\ g_{66}=\left(g_{\varphi_4}[\varphi_4]'\right)^2\cdot r^{-i\cdot s};\quad g_{77}=-\left(g_{\varphi_5}[\varphi_5]'\right)^2\cdot r^{-s}; \end{cases} \\
s=2\quad\begin{cases} g_{88}=\left(g_{\varphi_2}[\varphi_2]'\right)^2\cdot r^{i\cdot s};\quad g_{99}=\left(g_{\varphi_3}[\varphi_3]'\right)^2\cdot r^{s}; \\ g_{1010}=\left(g_{\varphi_4}[\varphi_4]'\right)^2\cdot r^{-i\cdot s};\quad g_{1111}=-\left(g_{\varphi_5}[\varphi_5]'\right)^2\cdot r^{-s}; \end{cases}
\end{gathered} \qquad (15)$$

Introducing the potential function k[r] again, this approach can be simplified as follows:

$$g^{n}_{\alpha\beta}=\begin{pmatrix} -c^2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & r^2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & r^2\cdot\sin^2\varphi_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{n-1n-1} \end{pmatrix}$$

$$\times F\left[f\left[t,r,\vartheta,\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5\right]\right];\quad n=8;\quad F[f]=f^{\frac{4}{n-2}}=f^{\frac{2}{3}}$$

$$f\left[t,r,\vartheta,\varphi_1,\varphi_2,\varphi_3,\varphi_4,\varphi_5\right]=f_t\left[t\right]\cdot f_r\left[r\right]\cdot f_\vartheta\left[\vartheta\right]\cdot\prod_{i=1}^{5}f_{\varphi_i}\left[\varphi_i\right];\tag{16}$$

$$g_{44}=\left(g_{\varphi_2}\left[\varphi_2\right]'\right)^2\cdot k[r]^{i\cdot s};\quad g_{55}=\left(g_{\varphi_3}\left[\varphi_3\right]'\right)^2\cdot k[r]^{s};$$

$$g_{66}=\left(g_{\varphi_4}\left[\varphi_4\right]'\right)^2\cdot k[r]^{-i\cdot s};\quad g_{77}=\left(g_{\varphi_5}\left[\varphi_5\right]'\right)^2\cdot k[r]^{-s}$$

Using the separation solution from above with:

$$\begin{gathered} f_{\varphi_i}\left[\varphi_i\right]=f_{\varphi_i}\left[g_{\varphi_i}\left[\varphi_i\right]\right]\equiv f_{\varphi_i}\left[g_{\varphi_i}\right]; \\ f_{\varphi_i}\left[g_{\varphi_i}\right]=C_{-i}\cdot e^{-i\cdot A_i\cdot g_{\varphi i}}+C_{+i}\cdot e^{+i\cdot A_i\cdot g_{\varphi i}}=C_{ci}\cdot\cos\left[A_i\cdot g_{\varphi_i}\right]+C_{si}\cdot\sin\left[A_i\cdot g_{\varphi_i}\right] \\ \text{especially}\quad(\text{classically}): \\ f_{\varphi 1}\left[g_{\varphi_1}\right]=C_{-1}\cdot e^{-i\cdot A_1\cdot g_{\varphi 1}}+C_{+1}\cdot e^{+i\cdot A_1\cdot g_{\varphi 1}}=C_{c1}\cdot\cos\left[\overbrace{A_1}^{\hat{=}m}\cdot g_{\varphi_1}\right]+C_{s1}\cdot\sin\left[\overbrace{A_1}^{\hat{=}m}\cdot g_{\varphi_1}\right] \end{gathered},\tag{17}$$

$$f_t\left[t\right]=C_{t1}\cdot\cos\left[c\cdot\sqrt{E}\cdot t\right]+C_{t2}\cdot\sin\left[c\cdot\sqrt{E}\cdot t\right],\tag{18}$$

$$f_\vartheta\left[\vartheta=\varphi_0\right]=C_{P\vartheta}\cdot P_L^{A_1}\left[\cos\left[\vartheta=\varphi_0\right]\right]+C_{Q\vartheta}\cdot Q_L^{A_1}\left[\cos\left[\vartheta=\varphi_0\right]\right],\tag{19}$$

with the associated Legendre polynomials $P_L^{A_1},Q_L^{A_1}$, and inserting everything into (14), we obtain the following differential equation in r:

$$\begin{gathered} R^*=0=\begin{pmatrix} \frac{L\cdot(L+1)}{r^2}+k[r]^{i\cdot s}\cdot A_2^2+k[r]^{s}\cdot A_3^2 \\ +k[r]^{-i\cdot s}\cdot A_4^2+k[r]^{-s}\cdot A_5^2-\overbrace{\Delta_{3D-sphere}[r]}^{2\frac{\partial}{r\cdot\partial r}+\frac{\partial^2}{\partial r^2}}-E^2 \end{pmatrix}\Psi \\ \Psi=\Psi\left[r,\vartheta=\varphi_0,\varphi=\varphi_1\right]=f_r\left[r\right]\cdot f_\vartheta\left[\vartheta=\varphi_0\right]\cdot f_\varphi\left[\varphi=\varphi_1\right] \\ \Rightarrow \\ 0=f_\vartheta\left[\vartheta\right]\cdot f\left[\varphi\right]\cdot\begin{pmatrix} \frac{L\cdot(L+1)}{r^2}+k[r]^{i\cdot s}\cdot A_2^2+k[r]^{s}\cdot A_3^2+k[r]^{-i\cdot s}\cdot A_4^2 \\ +k[r]^{-s}\cdot A_5^2-\frac{2\partial}{r\cdot\partial r}-\frac{\partial^2}{\partial r^2}-E^2 \end{pmatrix}f_r\left[r\right] \end{gathered}.\tag{20}$$

We see that for the choice of k[r] as:

$$k\left[r\right]=-\frac{1}{A_3^2}\left(\frac{V}{r}-E_P^2\right),\tag{21}$$

s=1 and $f_{\varphi_i}[\varphi_i]=\text{const}$ for i=2,4,5, we obtain:

$$0=f_{\vartheta}[\vartheta]\cdot f[\varphi]\cdot\left(\frac{L\cdot(L+1)}{r^2}-\frac{V}{r}+E_P^2-\frac{2\partial}{r\cdot\partial r}-\frac{\partial^2}{\partial r^2}-E\right)f_r[r]$$
$$\Rightarrow$$
$$\left(-\frac{L\cdot(L+1)}{r^2}+\frac{V}{r}+2\frac{\partial}{r\cdot\partial r}+\frac{\partial^2}{\partial r^2}\right)f_r[r]=\left(E_P^2-E\right)f_r[r] \quad (22)$$

and thus, the classical Schrödinger equation again (regarding its solution see text book literature (e.g. [4]) or the original papers from Schrödinger [5, 6]).

References Appendix

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