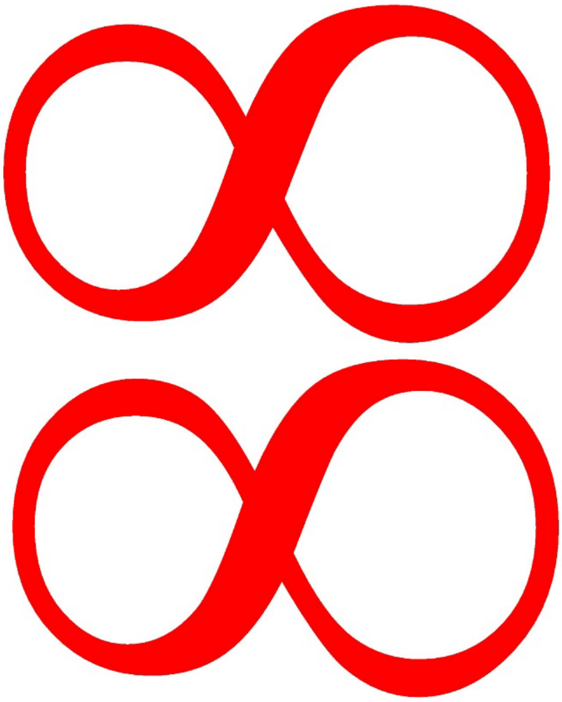
Infinite Orthogonal Dimensionality

Part II: Why the Dirac Theory



by

Dr. Norbert Schwarzer

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Infinite Orthogonal Dimensionality Part II: Why the Dirac Theory

By Dr. Norbert Schwarzer

1 Abstract

In part I [A] of this series we discussed the question of universal infinity and linearity. Thereby we already saw that there are numerous ways to achieve linearity in a quantum gravity environment.

In this short paper we are now investigating a linearization technique which directly leads to the Dirac theory. Our goal is the partial linearization of the quantum Einstein field equations by the means of a special transformation.

[A] N. Schwarzer, "Infinite Orthogonal Dimensionality - Part I: Why We Need Linearity and How Does This Make the Space-Time Jitter", 2025, a SIO science paper, www.siomec.de

2 Introduction

We take it that the reader is familiar with the equations resulting from a Hilbert variation [1] with respect to a scaled metric of the kind:

$$G_{\alpha\beta} = g_{\alpha\beta} \cdot F[f]. \tag{1}$$

If not, the necessary ingredients can be found in the appendix of this paper and our previous publications [2–8]. The reader will also need to study our extended Hamilton principle (also appendix) with the resulting field equations:

$$\begin{aligned} 0 &= \int_V d^n x \sqrt{-g} \left(R_{\kappa\lambda} \delta g^{\kappa\lambda} - R \cdot \left(\frac{1}{2} + H \right) g_{\kappa\lambda} \delta g^{\kappa\lambda} \right) \\ &= \int_V d^n x \sqrt{-g} \left(R_{\kappa\lambda} - R \cdot \left(\frac{1}{2} + H \right) g_{\kappa\lambda} \right) \delta g^{\kappa\lambda} \quad . \\ &\Rightarrow R_{\kappa\lambda} - R \cdot \left(\frac{1}{2} + H \right) g_{\kappa\lambda} = 0 \end{aligned} \tag{2}$$

Its expanded form in the case of a scaled metric (1) is given as follows:

$$0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}$$

$$=\left(\begin{array}{c}\left(R_{\alpha\beta}-\frac{F'}{2F}\left(\begin{array}{l}f_{,\alpha\beta}(n-2)+f_{,ab}g_{\alpha\beta}g^{ab}\\+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-\\f_{,\alpha}g^{ab}g_{\beta b,a}-f_{,\beta}g^{ab}g_{\alpha b,a}\\+f_{,d}g^{cd}\left(\begin{array}{l}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\+\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)\end{array}\right)+\frac{1}{4F^2}\left(\begin{array}{l}f_{,\alpha}\cdot f_{,\beta}(n-2)\left(3(F')^2-2FF''\right)\\+g_{\alpha\beta}f_{,c}f_{,d}g^{cd}\left((F')^2(4-n)-2FF''\right)\end{array}\right)\right)\\-\left(\begin{array}{c}R-\frac{F'}{2F}\left(\begin{array}{c}(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)\\-nf_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right)\\-(n-1)\frac{g^{ab}f_{,a}\cdot f_{,b}}{4F^2}\left(4FF''+(F')^2(n-6)\right)\end{array}\right)\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\end{array}\right)\delta G^{\alpha\beta}\quad. \tag{3}$$

While in (3) we kept the structure of the classical field equations from Einstein's General Theory of Relativity [9], with the tensorial and the scalar Ricci parts well separated, in some cases it is useful to use the following form:

$$0=\left(\begin{array}{c}2\frac{F}{F'}\cdot R_{\alpha\beta}-\left(\begin{array}{c}f_{,\alpha\beta}(n-2)+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-f_{,\alpha}g^{ab}g_{\beta b,a}\\-f_{,\beta}g^{ab}g_{\alpha b,a}+f_{,d}g^{cd}\left(g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}+\frac{1}{2}ng_{\alpha\beta,c}\right)\end{array}\right)\\-g_{\alpha\beta}\left(f_{,ab}g^{ab}+f_{,d}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right)\\+g_{\alpha\beta}\left(\left(\begin{array}{c}(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)\\-nf_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right)-2\frac{F}{F'}\cdot R\right)\cdot\left(\frac{1}{2}+H\right)\\+\frac{1}{2F\cdot F'}f_{,\alpha}\cdot f_{,\beta}(n-2)\left(3(F')^2-2FF''\right)\\+g_{\alpha\beta}\frac{1}{2F\cdot F'}f_{,a}f_{,b}g^{ab}\left(\begin{array}{c}(F')^2\left(4-n+(n-1)(n-6)\cdot\left(\frac{1}{2}+H\right)\right)\\+2FF''\left(2(n-1)\cdot\left(\frac{1}{2}+H\right)-1\right)\end{array}\right)\end{array}\right). \tag{4}$$

In [2–8] it was shown how certain settings of the so-called wrapping function F[f] would help us to get rid of some of the f-functional nonlinearities in (3). So, for instance, a setting of the kind:

$$F[f]=\begin{cases}C_F\cdot\left(f+C_f\right)^{\frac{4}{n-2}} & n\neq 2\\ C_F\cdot e^{f\cdot C_f} & n=2\end{cases} \tag{5}$$

gives us:

$$0=\left(\begin{array}{l}\left(\begin{array}{l}R_{\alpha\beta}-\dfrac{F'}{2F}\left(\begin{array}{l}f_{,\alpha\beta}(n-2)+f_{,ab}g_{\alpha\beta}g^{ab}\\+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-\\f_{,\alpha}g^{ab}g_{\beta b,a}-f_{,\beta}g^{ab}g_{\alpha b,a}\\+f_{,d}g^{cd}\left(\begin{array}{l}g_{\alpha c,\beta}-\dfrac{1}{2}ng_{\alpha c,\beta}-\dfrac{1}{2}ng_{\beta c,\alpha}\\+\dfrac{1}{2}ng_{\alpha\beta,c}+\dfrac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)\end{array}\right)\\+\dfrac{1}{4F^2}\left(\begin{array}{l}f_{,\alpha}\cdot f_{,\beta}(n-2)\left(3(F')^2-2FF''\right)\\+g_{\alpha\beta}f_{,c}f_{,d}g^{cd}\left((F')^2(4-n)-2FF''\right)\end{array}\right)\end{array}\right)\\-\left(R-\dfrac{F'}{2F}\left(\begin{array}{c}(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)\\-nf_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right)\right)\cdot\left(\dfrac{1}{2}+H\right)g_{\alpha\beta}\end{array}\right)\delta G^{\alpha\beta}, \quad (6)$$

while:

$$F[f]=C_{f1}\left(f+C_{f0}\right)^{-2}, \quad (7)$$

$$F[f]=\begin{cases}C_F\cdot\left(f+C_f\right)^{\frac{2}{n-2}} & n\neq 2\\ C_F\cdot e^{f\cdot C_f} & n=2\end{cases}, \quad (8)$$

$$F[f]=\begin{cases}C_F\cdot\left(f+C_f\right)^{\frac{4(2H(n-1)+n-2)}{(n-2)(2H(n-1)+n-3)}} & n\neq 2\\ C_F\cdot e^{f\cdot C_f} & n=2\cup n=\dfrac{3+2H}{1+2H}\end{cases}, \quad (9)$$

result in:

$$0=\left(\begin{array}{l}\left(\begin{array}{c}R_{\alpha\beta}-\dfrac{F'}{2F}\left(\begin{array}{c}f_{,\alpha\beta}(n-2)+f_{,ab}g_{\alpha\beta}g^{ab}+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)\\-f_{,\alpha}g^{ab}g_{\beta b,a}-f_{,\beta}g^{ab}g_{\alpha b,a}\\+f_{,d}g^{cd}\left(\begin{array}{l}g_{\alpha c,\beta}-\dfrac{1}{2}ng_{\alpha c,\beta}-\dfrac{1}{2}ng_{\beta c,\alpha}\\+\dfrac{1}{2}ng_{\alpha\beta,c}+\dfrac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)\end{array}\right)\\+\dfrac{1}{4F^2}g_{\alpha\beta}f_{,c}f_{,d}g^{cd}\left((F')^2(4-n)-2FF''\right)\end{array}\right)\\-\left(\begin{array}{l}R-\dfrac{F'}{2F}\left(\begin{array}{c}(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)\\-nf_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right)\\-(n-1)\dfrac{g^{ab}f_{,a}\cdot f_{,b}}{4F^2}\left(4FF''+(F')^2(n-6)\right)\end{array}\right)\cdot\left(\dfrac{1}{2}+H\right)g_{\alpha\beta}\end{array}\right)\delta G^{\alpha\beta}, \quad (10)$$

$$0=\left(\begin{array}{c}\left(\begin{array}{c} R_{\alpha\beta}-\dfrac{F'}{2F}\left(\begin{array}{l} f_{,\alpha\beta}(n-2)+f_{,ab}g_{\alpha\beta}g^{ab} \\ +f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)- \\ f_{,\alpha}g^{ab}g_{\beta b,a}-f_{,\beta}g^{ab}g_{\alpha b,a} \\ +f_{,d}g^{cd}\left(\begin{array}{c} g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha} \\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab} \end{array}\right) \end{array}\right) \\ +\dfrac{1}{4F^{2}}f_{,\alpha}\cdot f_{,\beta}(n-2)\left(3(F')^{2}-2FF''\right) \end{array}\right) \\ -\left(\begin{array}{c} R-\dfrac{F'}{2F}\left(\begin{array}{c}(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right) \\ -nf_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right) \\ -(n-1)\dfrac{g^{ab}f_{,a}\cdot f_{,b}}{4F^{2}}\left(4FF''+(F')^{2}(n-6)\right) \end{array}\right)\cdot\left(\dfrac{1}{2}+H\right)g_{\alpha\beta}\end{array}\right)\delta G^{\alpha\beta}, \tag{11}$$

and:

$$0=\left(\begin{array}{c} 2\dfrac{F}{F'}\cdot R_{\alpha\beta}-\left(\begin{array}{c} f_{,\alpha\beta}(n-2)+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-f_{,\alpha}g^{ab}g_{\beta b,a} \\ -f_{,\beta}g^{ab}g_{\alpha b,a}+f_{,d}g^{cd}\left(g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}+\frac{1}{2}ng_{\alpha\beta,c}\right)\end{array}\right) \\ -g_{\alpha\beta}\left(f_{,ab}g^{ab}+f_{,d}g^{cd}\dfrac{1}{2}g_{ab,c}g^{ab}\right) \\ +g_{\alpha\beta}\left(\left((n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)-nf_{,d}g^{cd}g^{ab}g_{ac,b}\right)-2\dfrac{F}{F'}\cdot R\right)\cdot\left(\dfrac{1}{2}+H\right) \\ +\dfrac{1}{2F\cdot F'}f_{,\alpha}\cdot f_{,\beta}(n-2)\left(3(F')^{2}-2FF''\right)\end{array}\right), \tag{12}$$

respectively.

3 A Strange Base-Vector and / or Dirac(!?!)-Linearity

So far (c.f. [3–8]), we have achieved linearity via certain restrictions to the variation or introduced wrapping functions for the metric volume and/or the kernel of the variational integral. Now we want to consider a possibility of obtaining linearity for the quantum field equations directly through the solution for the functions F and f without any restrictions to the variation or the introduction of kernel factors. Hence, our starting point shall be the field equation of the form (3), where we now seek to find linearity with the following ansatz for F:

$$F=F[f]=F\left[f\left[\sum_{i=0}^{n-1}C_{i}\cdot x_{i}\right]\right]. \tag{13}$$

Thereby, for the reason of simplicity, we assume a metric of constants, which simplifies equation (3):

$$0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}$$

$$=\left(\begin{pmatrix}-\frac{F'}{2F}\left(f_{,\alpha\beta}(n-2)+f_{,ab}g_{\alpha\beta}g^{ab}\right)\\ +\frac{1}{4F^{2}}\begin{pmatrix}f_{,\alpha}\cdot f_{,\beta}(n-2)\left(3(F')^{2}-2FF''\right)\\ +g_{\alpha\beta}f_{,c}f_{,d}g^{cd}\left((F')^{2}(4-n)-2FF''\right)\end{pmatrix}\end{pmatrix}\\ -\begin{pmatrix}-\frac{F'}{F}(n-1)g^{ab}f_{,ab}\\ -(n-1)\frac{g^{ab}f_{,a}\cdot f_{,b}}{4F^{2}}\left(4FF''+(F')^{2}(n-6)\right)\end{pmatrix}\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\right)\delta G^{\alpha\beta}. \tag{14}$$

We immediately see that with the condition:

$$C_i\cdot C_j=g_{ij} \tag{15}$$

and F chosen in accordance with (5), equation (14) simplifies to:

$$0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}$$

$$=\left(\begin{pmatrix}-\frac{F'}{F}g_{\alpha\beta}f''(n-1)\\ +\frac{1}{4F^{2}}\begin{pmatrix}g_{\alpha\beta}(f')^{2}(n-2)\left(3(F')^{2}-2FF''\right)\\ +g_{\alpha\beta}n(f')^{2}\left((F')^{2}(4-n)-2FF''\right)\end{pmatrix}\end{pmatrix}\\ +n\cdot\frac{F'}{F}(n-1)f''\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\right)\delta G^{\alpha\beta}$$

$$=\left(\begin{pmatrix}-\frac{F'}{F}g_{\alpha\beta}f''(n-1)\\ +\frac{g_{\alpha\beta}(f')^{2}}{4F^{2}}\left((n-2)\left(3(F')^{2}-2FF''\right)+n\left((F')^{2}(4-n)-2FF''\right)\right)\end{pmatrix}\\ +n\cdot\frac{F'}{F}(n-1)f''\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\right)\delta G^{\alpha\beta}$$

$$=\begin{pmatrix}\frac{F'}{F}f''\left(n\cdot\left(\frac{1}{2}+H\right)-1\right)g_{\alpha\beta}(n-1)\\ -\frac{g_{\alpha\beta}(f')^{2}}{4F^{2}}(n-1)\left(4FF''+(F')^{2}(n-6)\right)\end{pmatrix}\delta G^{\alpha\beta}$$

$$=\left(\frac{F'}{F}f''\left(n\cdot\left(\frac{1}{2}+H\right)-1\right)g_{\alpha\beta}(n-1)\right)\delta G^{\alpha\beta}. \tag{16}$$

As with our extreme simplicity chosen here, we would now have to demand f to just be:

$$f = f[x_i] = \sum_{i=0}^{n-1} C_i \cdot x_i \tag{17}$$

and our solution does not seem to be of much practical use, but the interesting aspect is that for metrics like the Minkowski metric, we have to realize that the C_i cannot just be ordinary numbers. In fact, we would require certain matrixes in order to fulfill conditions like these:

$$C_i \cdot C_j = \delta_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

We can easily see that the so-called base vectors $\mathbf{e}_i$ would do the job, because we have:

$$\mathbf{e}_i \cdot \mathbf{e}_j = g_{ij}. \tag{19}$$

Surprisingly, it would also be matrices like the Dirac matrices that could help us here. As an example we consider the space with n=4. There the Dirac matrices could be given as follows (all empty slots are zeros):

$$\begin{gathered} \gamma^0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}; \quad \gamma^1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \\ \gamma^2 = \begin{pmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{pmatrix}; \quad \gamma^3 = \begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}; \quad I = \delta_{ij} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \end{gathered}. \tag{20}$$

As the metric tensor is connected with these matrices in the following way:

$$g^{\alpha\beta} \cdot I = \frac{\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha}{2}; \quad g_{\alpha\beta} \cdot I = \frac{\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha}{2}, \tag{21}$$

we have to make the derivatives in (14) symmetric, which leads us to:

$$0 = \left(\begin{array}{c} \left(\begin{array}{c} -\frac{F'}{2F} \left(\frac{f_{,\alpha\beta} + f_{,\beta\alpha}}{2} (n-2) + \frac{f_{,ab} + f_{,ba}}{2} g_{\alpha\beta} g^{ab} \right) \\ + \frac{1}{4F^2} \left(\begin{array}{c} \frac{f_{,\alpha} f_{,\beta} + f_{,\beta} f_{,\alpha}}{2} (n-2) \left(3(F')^2 - 2FF'' \right) \\ + g_{\alpha\beta} \frac{f_{,c} f_{,d} + f_{,d} f_{,c}}{2} g^{cd} \left((F')^2 (4-n) - 2FF'' \right) \end{array} \right) \end{array} \right) \\ - \left(\begin{array}{c} -\frac{F'}{F} (n-1) g^{ab} \frac{f_{,ab} + f_{,ba}}{2} \\ -(n-1) \frac{g^{ab}}{4F^2} \frac{f_{,a} f_{,b} + f_{,b} f_{,a}}{2} \left(4FF'' + (F')^2 (n-6) \right) \end{array} \right) \cdot \left(\frac{1}{2} + H \right) g_{\alpha\beta} \end{array} \right) \delta G^{\alpha\beta} \tag{22}$$

and the corresponding metric volume function:

$$f = f\left[x_i\right] = \sum_{i=0}^{n-1} \gamma_i \cdot x_i \,. \tag{23}$$

3.1 Towards a Generalization of the Idea → Transformers Linearity

Admittedly, the new technique seems to be a bit unwieldy in terms of writing volume and object sizes, but it is pretty straightforward and might provide some advantages for the numerical handling of complex tasks and complicated geometries as they occur in connection with complex systems like internally structured fluids, for instance.

In this subsection we therefore intend to only roughly investigate the idea of using a transformation similar to the one discussed above in order to achieve linearity for the quantum gravity field equations… at least for the wave functions. Thereby we will concentrate on the effect of the various versions of the field equations and how the transformation could be applied. In order to move on quickly, we will at first be a bit “sloppy” with respect to the rigor of the incorporation of the technique. Later in this paper, we will then investigate the method with more intensity and accuracy.

3.1.1 The Introduction of Metric Separation Matrices

When aiming towards a generalization of the approach from above, we might like to introduce the following transformations:

$$\begin{gathered}
f_{,\alpha} \to C_\alpha^i f_{,i} \\
f_{,\alpha} f_{,\beta} \to C_\alpha^i C_\beta^j f_{,i} f_{,j} = g_{\alpha\beta} C^{ij} f_{,i} f_{,j} \\
f_{,\alpha\beta} \to C_\alpha^i A_\beta^j f_{,ij} = g_{\alpha\beta} B^{ij} f_{,ij} \xrightarrow{C_\alpha^i A_\beta^j f_{,ij} = C_\alpha^i C_\beta^j f_{,ij}} C_\alpha^i C_\beta^j f_{,ij} = g_{\alpha\beta} C^{ij} f_{,ij} \,. \\
f_{,a} f_{,b} g^{ab} \to C_a^i C_b^j f_{,i} f_{,j} = g^{ab} g_{ab} C^{ij} f_{,i} f_{,j} = n \cdot C^{ij} f_{,i} f_{,j} \\
\left\{ \begin{gathered}
f_{,ab} g^{ab} \to g^{ab} C_a^i A_b^j f_{,ij} = g^{ab} g_{ab} B^{ij} f_{,ij} = n \cdot B^{ij} f_{,ij} \\
\xrightarrow{C_\alpha^i A_\beta^j f_{,ij} = C_\alpha^i C_\beta^j f_{,ij}} \\
g^{ab} C_a^i C_b^j f_{,ij} = g^{ab} g_{ab} C^{ij} f_{,ij} = n \cdot C^{ij} f_{,ij}
\end{gathered} \right\}
\end{gathered} \tag{24}$$

Thereby we should point out that the arrow in the equations above should be understood as follows:

$$\begin{gathered}
f_{,\alpha} = C_\alpha^i \psi_{,i} \\
f_{,\alpha} f_{,\beta} = C_\alpha^i C_\beta^j \psi_{,i} \psi_{,j} = g_{\alpha\beta} C^{ij} \psi_{,i} \psi_{,j} \\
f_{,\alpha\beta} = C_\alpha^i A_\beta^j \psi_{,ij} = g_{\alpha\beta} B^{ij} \psi_{,ij} \xrightarrow{C_\alpha^i A_\beta^j \psi_{,ij} = C_\alpha^i C_\beta^j \psi_{,ij}} C_\alpha^i C_\beta^j \psi_{,ij} = g_{\alpha\beta} C^{ij} \psi_{,ij} \,, \\
f_{,a} f_{,b} g^{ab} = C_a^i C_b^j \psi_{,i} \psi_{,j} = g^{ab} g_{ab} C^{ij} \psi_{,i} \psi_{,j} = n \cdot C^{ij} \psi_{,i} \psi_{,j} \\
\left\{ \begin{gathered}
f_{,ab} g^{ab} = g^{ab} C_a^i A_b^j \psi_{,ij} = g^{ab} g_{ab} B^{ij} \psi_{,ij} = n \cdot B^{ij} \psi_{,ij} \\
\xrightarrow{C_\alpha^i A_\beta^j \psi_{,ij} = C_\alpha^i C_\beta^j \psi_{,ij}} \\
g^{ab} C_a^i C_b^j \psi_{,ij} = g^{ab} g_{ab} C^{ij} \psi_{,ij} = n \cdot C^{ij} \psi_{,ij}
\end{gathered} \right\}
\end{gathered} \tag{25}$$

and that our settings above are definitively too simple to truly be of practical relevance. We would probably end up in no more generality and / or flexibility than we have with (23), but we will keep the brevity of (24) and in fact mean more something like this:

$$
\begin{gathered}
\mathrm{I}\cdot f_{,\alpha}\to\vec{\vec{C}}_{\alpha}^{i}\vec{f}_{,i}\\
\mathrm{I}\cdot\mathrm{I}\cdot f_{,\alpha}f_{,\beta}\to\vec{\vec{C}}_{\alpha}^{i}\vec{\vec{C}}_{\beta}^{j}\vec{f}_{,i}\vec{f}_{,j}=g_{\alpha\beta}\vec{\vec{C}}^{ij}\vec{f}_{,i}\vec{f}_{,j}\\
\mathrm{I}\cdot\mathrm{I}\cdot f_{,\alpha\beta}\to\vec{\vec{C}}_{\alpha}^{i}\vec{\vec{A}}_{\beta}^{j}\vec{f}_{,ij}=g_{\alpha\beta}\vec{\vec{B}}^{ij}\vec{f}_{,ij}\xrightarrow{\vec{\vec{C}}_{\alpha}^{i}\vec{\vec{A}}_{\beta}^{j}\vec{f}_{,ij}=\vec{\vec{C}}_{\alpha}^{i}\vec{\vec{C}}_{\beta}^{j}\vec{f}_{,ij}}\vec{\vec{C}}_{\alpha}^{i}\vec{\vec{C}}_{\beta}^{j}\vec{f}_{,ij}=\mathrm{I}\cdot g_{\alpha\beta}\vec{\vec{C}}^{ij}\vec{f}_{,ij}\,,\\
\mathrm{I}\cdot\mathrm{I}\cdot f_{,a}f_{,b}g^{ab}\to\vec{\vec{C}}_{a}^{i}\vec{\vec{C}}_{b}^{j}\vec{f}_{,i}\vec{f}_{,j}=g^{ab}g_{ab}\mathrm{I}\cdot\vec{\vec{C}}^{ij}\vec{f}_{,i}\vec{f}_{,j}=n\cdot\mathrm{I}\cdot\vec{\vec{C}}^{ij}\vec{f}_{,i}\vec{f}_{,j}\\
\left\{\begin{array}{c}
\mathrm{I}\cdot\mathrm{I}\cdot f_{,ab}g^{ab}\to g^{ab}\vec{\vec{C}}_{a}^{i}\vec{\vec{A}}_{b}^{j}\vec{f}_{,ij}=g^{ab}g_{ab}\mathrm{I}\cdot\vec{\vec{B}}^{ij}\vec{f}_{,ij}=n\cdot\mathrm{I}\cdot\vec{\vec{B}}^{ij}\vec{f}_{,ij}\\
\xrightarrow{\vec{\vec{C}}_{\alpha}^{i}\vec{\vec{A}}_{\beta}^{j}\vec{f}_{,ij}=\vec{\vec{C}}_{\alpha}^{i}\vec{\vec{C}}_{\beta}^{j}\vec{f}_{,ij}}\\
g^{ab}\vec{\vec{C}}_{a}^{i}\vec{\vec{C}}_{b}^{j}\vec{f}_{,ij}=g^{ab}g_{ab}\mathrm{I}\cdot\vec{\vec{C}}^{ij}\vec{f}_{,ij}=n\cdot\mathrm{I}\cdot\vec{\vec{C}}^{ij}\vec{f}_{,ij}
\end{array}\right\}
\end{gathered}
\tag{26}
$$

where we immediately see that each component of the objects C_{α}^{i} are matrices and the functions f have to become vectors (spinors). So, for brevity, throughout this paper, the objects C_{α}^{i} have matrix components, but we refrain from explicitly pointing this out everywhere and apply the shorter forms (24) instead.

Now we can rewrite equation (14) as follows (thereby always remembering that the components of the $C_{\alpha}^{i},C^{ij},A_{\beta}^{j},B^{ij}$ are no simple numbers but complex objects $\vec{\vec{C}}_{\alpha}^{i},\vec{\vec{C}}^{ij},\vec{\vec{A}}_{\beta}^{j},\vec{\vec{B}}^{ij}$):

$$
\begin{aligned}
0&=g_{\alpha\beta}\left(\begin{array}{c}
\left(\begin{array}{c}
-\frac{F'}{2F}\left(C^{ij}f_{,ij}(n-2)+n\cdot C^{ij}f_{,ij}\right)\\
+\frac{1}{4F^{2}}\left(\begin{array}{c}C^{ij}f_{,i}f_{,j}(n-2)\left(3(F')^{2}-2FF''\right)\\+n\cdot C^{ij}f_{,i}f_{,j}\left((F')^{2}(4-n)-2FF''\right)\end{array}\right)
\end{array}\right)\\
-(n-1)\left(-\frac{F'}{F}n\cdot C^{ij}f_{,ij}-\frac{n\cdot C^{ij}f_{,i}f_{,j}}{4F^{2}}\left(4FF''+(F')^{2}(n-6)\right)\right)\cdot\left(\frac{1}{2}+H\right)
\end{array}\right)\delta G^{\alpha\beta}\\
&=g_{\alpha\beta}\left(\begin{array}{c}
\left(\begin{array}{c}
-\frac{F'}{F}(n-1)C^{ij}f_{,ij}\\
+\frac{C^{ij}f_{,i}f_{,j}}{4F^{2}}\left((n-2)\left(3(F')^{2}-2FF''\right)+n\cdot\left((F')^{2}(4-n)-2FF''\right)\right)
\end{array}\right)\\
-n\cdot(n-1)\left(-\frac{F'}{F}C^{ij}f_{,ij}-\frac{C^{ij}f_{,i}f_{,j}}{4F^{2}}\left(4FF''+(F')^{2}(n-6)\right)\right)\cdot\left(\frac{1}{2}+H\right)
\end{array}\right)\delta G^{\alpha\beta}\\
&=g_{\alpha\beta}(n-1)\left(\begin{array}{c}
\left(\begin{array}{c}
-\frac{F'}{F}C^{ij}f_{,ij}\\
+\frac{C^{ij}f_{,i}f_{,j}}{4F^{2}(n-1)}\left((F')^{2}\left(3(n-2)+n\cdot(4-n)\right)-4FF''(n-1)\right)
\end{array}\right)\\
+n\cdot\left(\frac{F'}{F}C^{ij}f_{,ij}+\frac{C^{ij}f_{,i}f_{,j}}{4F^{2}}\left(4FF''+(F')^{2}(n-6)\right)\right)\cdot\left(\frac{1}{2}+H\right)
\end{array}\right)\delta G^{\alpha\beta}\;.
\end{aligned}
\tag{27}
$$

We see that also the middle line results in the same equation for F[f] as we already have in the last line (which represents the Ricci scalar part of the quantum Einstein field equations), namely:

$$
\begin{aligned}
0 &= g_{\alpha\beta}(n-1)\begin{pmatrix} \begin{pmatrix} -\dfrac{F'}{F}C^{ij}f_{,ij} \\ +\dfrac{C^{ij}f_{,i}f_{,j}}{4F^2(n-1)}\Big((F')^2\big((n-1)(6-n)\big)-4FF''(n-1)\Big) \end{pmatrix} \\ +n\cdot\left(\dfrac{F'}{F}C^{ij}f_{,ij}+\dfrac{C^{ij}f_{,i}f_{,j}}{4F^2}\Big(4FF''+(F')^2(n-6)\Big)\right)\cdot\left(\dfrac{1}{2}+H\right) \end{pmatrix}\delta G^{\alpha\beta} \\
&= g_{\alpha\beta}(n-1)\begin{pmatrix} \dfrac{C^{ij}f_{,i}f_{,j}}{4F^2}\Big((F')^2(6-n)-4FF''\Big)-\dfrac{F'}{F}C^{ij}f_{,ij} \\ +n\cdot\left(\dfrac{F'}{F}C^{ij}f_{,ij}+\dfrac{C^{ij}f_{,i}f_{,j}}{4F^2}\Big(4FF''+(F')^2(n-6)\Big)\right)\cdot\left(\dfrac{1}{2}+H\right) \end{pmatrix}\delta G^{\alpha\beta} \quad . \qquad (28) \\
&= g_{\alpha\beta}(n-1)\begin{pmatrix} -\dfrac{C^{ij}f_{,i}f_{,j}}{4F^2}\Big(4FF''+(F')^2(n-6)\Big)-\dfrac{F'}{F}C^{ij}f_{,ij} \\ +n\cdot\left(\dfrac{F'}{F}C^{ij}f_{,ij}+\dfrac{C^{ij}f_{,i}f_{,j}}{4F^2}\Big(4FF''+(F')^2(n-6)\Big)\right)\cdot\left(\dfrac{1}{2}+H\right) \end{pmatrix}\delta G^{\alpha\beta}
\end{aligned}
$$

With F[f] being chosen in accordance with (5), we would consequently get rid of the nonlinear terms and obtain:

$$
0 = g_{\alpha\beta}(n-1)\frac{F'}{F}C^{ij}f_{,ij}\left(n\cdot\left(\frac{1}{2}+H\right)-1\right)\delta G^{\alpha\beta} \,. \qquad (29)
$$

Applying the recipe on the full equation (3) without any restrictions to the metric gives us:

$$
\begin{aligned}
0 &= \left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}} \\
&= \begin{pmatrix} \begin{pmatrix} R_{\alpha\beta}-\dfrac{F'}{2F}\begin{pmatrix} g_{\alpha\beta}B^{ij}f_{,ij}(n-2)+n\cdot B^{ij}f_{,ij}g_{\alpha\beta} \\ +C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a}-C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a} \\ +C^{i}_{d}f_{,i}g^{cd}\begin{pmatrix} g_{\alpha c,\beta}-\dfrac{1}{2}ng_{\alpha c,\beta}-\dfrac{1}{2}ng_{\beta c,\alpha} \\ +\dfrac{1}{2}ng_{\alpha\beta,c}+\dfrac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab} \end{pmatrix} \end{pmatrix} \\ +\dfrac{g_{\alpha\beta}}{4F^2}\begin{pmatrix} C^{ij}f_{,i}f_{,j}(n-2)\Big(3(F')^2-2FF''\Big) \\ +n\cdot C^{ij}f_{,i}f_{,j}\Big((F')^2(4-n)-2FF''\Big) \end{pmatrix} \end{pmatrix} \\ -\begin{pmatrix} R-\dfrac{F'}{2F}\begin{pmatrix} (n-1)\Big(2n\cdot B^{ij}f_{,ij}+C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\Big) \\ -nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b} \end{pmatrix} \\ -(n-1)\dfrac{n\cdot C^{ij}f_{,i}f_{,j}}{4F^2}\Big(4FF''+(F')^2(n-6)\Big) \end{pmatrix}\cdot\left(\dfrac{1}{2}+H\right)g_{\alpha\beta} \end{pmatrix}\delta G^{\alpha\beta} \quad , \qquad (30)
\end{aligned}
$$

which simplifies to:

$$
0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}
$$

$$
=\left(\begin{array}{l}
R_{\alpha\beta}-\dfrac{F'}{2F}\left(\begin{array}{c}
2g_{\alpha\beta}B^{ij}f_{,ij}(n-1)+C^{i}_{d}f_{,i}g^{cd}\left(\begin{array}{l}
g_{\alpha c,\beta}-\dfrac{1}{2}ng_{\alpha c,\beta}-\dfrac{1}{2}ng_{\beta c,\alpha}\\
+\dfrac{1}{2}ng_{\alpha\beta,c}+\dfrac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}
\end{array}\right)\\
+C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a}-C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a}
\end{array}\right)\\
\qquad -(n-1)\dfrac{g_{\alpha\beta}}{4F^{2}}C^{ij}f_{,i}f_{,j}\left(4FF''+(F')^{2}(n-6)\right)\\
-\left(\begin{array}{l}
R-\dfrac{F'}{2F}\left(\begin{array}{c}
(n-1)\left(2n\cdot B^{ij}f_{,ij}+C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right)\\
-nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b}
\end{array}\right)\\
-(n-1)\dfrac{n\cdot C^{ij}f_{,i}f_{,j}}{4F^{2}}\left(4FF''+(F')^{2}(n-6)\right)
\end{array}\right)\cdot\left(\dfrac{1}{2}+H\right)g_{\alpha\beta}
\end{array}\right)\delta G^{\alpha\beta}\quad . \qquad (31)
$$

We realize that with the introduction of our “metric matrices” (26), here used without always pointing out the matrix character of each component and abbreviating via (24), our quantum gravity equations simplified with respect to the nonlinear terms. Our little Dirac-like trick allowed us to collect all f-nonlinearity in just one term, namely (see box in the last line):

$$
0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}
$$

$$
=\left(\begin{array}{l}
R_{\alpha\beta}-\dfrac{F'}{2F}\left(\begin{array}{c}
2g_{\alpha\beta}B^{ij}f_{,ij}(n-1)+C^{i}_{d}f_{,i}g^{cd}\left(\begin{array}{l}
g_{\alpha c,\beta}-\dfrac{1}{2}ng_{\alpha c,\beta}-\dfrac{1}{2}ng_{\beta c,\alpha}\\
+\dfrac{1}{2}ng_{\alpha\beta,c}+\dfrac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}
\end{array}\right)\\
+C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a}-C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a}
\end{array}\right)\\
-\left(R-\dfrac{F'}{2F}\left(\begin{array}{c}
(n-1)\left(2n\cdot B^{ij}f_{,ij}+C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right)\\
-nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b}
\end{array}\right)\right)\cdot\left(\dfrac{1}{2}+H\right)g_{\alpha\beta}\\
+\boxed{(n-1)\dfrac{g_{\alpha\beta}}{4F^{2}}C^{ij}f_{,i}f_{,j}\left(4FF''+(F')^{2}(n-6)\right)\left(n\cdot\left(\dfrac{1}{2}+H\right)-1\right)}
\end{array}\right)\delta G^{\alpha\beta}\quad . \qquad (32)
$$

From here now, linearization of the quantum field equations with respect to the wave function f is straightforward.

3.1.1.1 The Metric Klein-Gordon Equation

3.1.1.1.1 Full Tensor and Scalar Separation

Using condition (5) again yields:

$$0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}$$

$$=\left(\begin{array}{c}\left(R_{\alpha\beta}-\frac{F'}{2F}\left(\begin{array}{c}2g_{\alpha\beta}B^{ij}f_{,ij}\left(n-1\right)+C^{i}_{d}f_{,i}g^{cd}\left(\begin{array}{c}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\+\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)\\+C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a}-C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a}\end{array}\right)\right)\\-\left(R-\frac{F'}{2F}\left(\begin{array}{c}\left(n-1\right)\left(2n\cdot B^{ij}f_{,ij}+C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right)\\-nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b}\end{array}\right)\right)\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\end{array}\right)\delta G^{\alpha\beta}\qquad(33)$$

Now we distinguish again rigorously between scalar terms times the metric tensor $g_{\alpha\beta}$ and tensor parts:

$$0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}$$

$$=\left(\begin{array}{c}R_{\alpha\beta}-\frac{F'}{2F}\left(\begin{array}{c}C^{i}_{d}f_{,i}g^{cd}\left(g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}+\frac{1}{2}ng_{\alpha\beta,c}\right)\\+C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a}-C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a}\end{array}\right)\\-g_{\alpha\beta}\left(\begin{array}{c}\left(R-\frac{F'}{2F}\left(\begin{array}{c}\left(n-1\right)\left(2n\cdot B^{ij}f_{,ij}+C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right)\\-nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b}\end{array}\right)\right)\cdot\left(\frac{1}{2}+H\right)\\-\frac{F'}{2F}\left(2B^{ij}f_{,ij}\left(n-1\right)+C^{i}_{d}f_{,i}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right)\end{array}\right)\end{array}\right)\delta G^{\alpha\beta}\qquad(34)$$

We demand:

$$0=\left(\begin{array}{c}\left(\frac{2F}{F'}R-\left(\left(n-1\right)\left(2n\cdot B^{ij}f_{,ij}+C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right)-nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b}\right)\right)\cdot\left(\frac{1}{2}+H\right)\\-\left(2B^{ij}f_{,ij}\left(n-1\right)+C^{i}_{d}f_{,i}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right)\end{array}\right)\qquad(35)$$

and use the resulting solution for f to find the metric tensor via the remaining field equations:

$$0=R_{\alpha\beta}-\frac{F'}{2F}\left(\begin{array}{c}C^{i}_{d}f_{,i}g^{cd}\left(g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}+\frac{1}{2}ng_{\alpha\beta,c}\right)\\+C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a}-C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a}\end{array}\right).\qquad(36)$$

3.1.1.1.2 Ricci Tensor and Ricci Scalar Separation

Starting directly with (33), we demand:

$$0=R-\frac{F'}{2F}\left(\left(n-1\right)\left(2n\cdot B^{ij}f_{,ij}+C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right)-nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b}\right)\qquad(37)$$

for the determination of the wave function f, and subsequently obtain the following equation for the determination of the metric tensor:

$$0 = R_{\alpha\beta} - \frac{F'}{2F}\left(\begin{array}{c} 2g_{\alpha\beta}B^{ij}f_{,ij}\left(n-1\right) + C^{i}_{d}f_{,i}g^{cd}\left(\begin{array}{c} g_{\alpha c,\beta} - \frac{1}{2}ng_{\alpha c,\beta} - \frac{1}{2}ng_{\beta c,\alpha} \\ + \frac{1}{2}ng_{\alpha\beta,c} + \frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab} \end{array}\right) \\ + C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha} - g_{\beta\alpha,b}\right) - C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a} - C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a} \end{array}\right). \tag{38}$$

We see that in this case the parameter *H* plays no role, and it would not matter whether we accept the classical Hamilton extremal principle or rather keep it undogmatic and general as the "principle of the ever-jittering fulcrum" [13].

3.1.1.1.3 Gravity and Matter Separation

Following the classical structure of the Einstein field equations, we could interpret everything not being contained by the classical vacuum field equations, which—in our case with the generalized Hamilton principle—would be:

$$0 = \left(R_{\alpha\beta} - R\cdot\left(\frac{1}{2} + H\right)g_{\alpha\beta}\right), \tag{39}$$

as matter and separate consequently as follows:

$$0 = \left(\begin{array}{c} R_{\alpha\beta} - R\cdot\left(\frac{1}{2} + H\right)g_{\alpha\beta} \\ -\frac{F'}{2F}\left(\begin{array}{c} 2g_{\alpha\beta}B^{ij}f_{,ij}\left(n-1\right) + C^{i}_{d}f_{,i}g^{cd}\left(\begin{array}{c} g_{\alpha c,\beta} - \frac{1}{2}ng_{\alpha c,\beta} - \frac{1}{2}ng_{\beta c,\alpha} \\ + \frac{1}{2}ng_{\alpha\beta,c} + \frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab} \end{array}\right) \\ + C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha} - g_{\beta\alpha,b}\right) - C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a} - C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a} \end{array}\right) \\ + \frac{F'}{2F}\left(\begin{array}{c} \left(n-1\right)\left(2n\cdot B^{ij}f_{,ij} + C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right) \\ -nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b} \end{array}\right)\cdot\left(\frac{1}{2} + H\right)g_{\alpha\beta} \end{array}\right). \tag{40}$$

Now we demand:

$$0 = \left(n-1\right)\left(2n\cdot B^{ij}f_{,ij} + C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right) - nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b} \tag{41}$$

for the determination of the wave function f and subsequently obtain the field equations:

$$0 = \left(\begin{array}{c} R_{\alpha\beta} - R\cdot\left(\frac{1}{2} + H\right)g_{\alpha\beta} \\ -\frac{F'}{2F}\left(\begin{array}{c} 2g_{\alpha\beta}B^{ij}f_{,ij}\left(n-1\right) + C^{i}_{d}f_{,i}g^{cd}\left(\begin{array}{c} g_{\alpha c,\beta} - \frac{1}{2}ng_{\alpha c,\beta} - \frac{1}{2}ng_{\beta c,\alpha} \\ + \frac{1}{2}ng_{\alpha\beta,c} + \frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab} \end{array}\right) \\ + C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha} - g_{\beta\alpha,b}\right) - C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a} - C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a} \end{array}\right) \end{array}\right). \tag{42}$$

The latter equations, which are, obviously, field equations with matter, determine the metric tensor.

As being interested in completely reproducing the classical field equations, resulting from the Einstein-Hilbert action with the extremal principle with the vacuum part:

$$0=\left(R_{\alpha\beta}-\frac{1}{2}R\cdot g_{\alpha\beta}\right), \tag{43}$$

we can separate as follows:

$$0=\left(\begin{array}{c} R_{\alpha\beta}-\frac{1}{2}R\cdot g_{\alpha\beta} \\ -\frac{F'}{2F}\left(\begin{array}{c} 2g_{\alpha\beta}B^{ij}f_{,ij}\left(n-1\right)+C_{d}^{i}f_{,i}g^{cd}\left(\begin{array}{c} g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha} \\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab} \end{array}\right) \\ +C_{a}^{i}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C_{\alpha}^{i}f_{,i}g^{ab}g_{\beta b,a}-C_{\beta}^{i}f_{,i}g^{ab}g_{\alpha b,a} \end{array}\right) \\ +\left(\frac{F'}{2F}\left(\begin{array}{c} \left(n-1\right)\left(2n\cdot B^{ij}f_{,ij}+C_{d}^{i}f_{,i}g^{cd}g^{ab}g_{ab,c}\right) \\ -nC_{d}^{i}f_{,i}g^{cd}g^{ab}g_{ac,b} \end{array}\right)\cdot\left(\frac{1}{2}+H\right)-R\cdot H\right)g_{\alpha\beta} \end{array}\right). \tag{44}$$

Demanding the determination of the wave function f via the linear scalar equation:

$$0=\frac{F'}{2F}\left(\begin{array}{c} \left(n-1\right)\left(2n\cdot B^{ij}f_{,ij}+C_{d}^{i}f_{,i}g^{cd}g^{ab}g_{ab,c}\right) \\ -nC_{d}^{i}f_{,i}g^{cd}g^{ab}g_{ac,b} \end{array}\right)\cdot\left(\frac{1}{2}+H\right)-R\cdot H, \tag{45}$$

leaves us with the following matter field equation:

$$0=\left(\begin{array}{c} R_{\alpha\beta}-\frac{1}{2}R\cdot g_{\alpha\beta} \\ -\frac{F'}{2F}\left(\begin{array}{c} 2g_{\alpha\beta}B^{ij}f_{,ij}\left(n-1\right)+C_{d}^{i}f_{,i}g^{cd}\left(\begin{array}{c} g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha} \\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab} \end{array}\right) \\ +C_{a}^{i}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C_{\alpha}^{i}f_{,i}g^{ab}g_{\beta b,a}-C_{\beta}^{i}f_{,i}g^{ab}g_{\alpha b,a} \end{array}\right) \end{array}\right), \tag{46}$$

where obviously the following term could be interpreted as the classical matter term:

$$0=-\frac{F'}{2F}\left(\begin{array}{c} 2g_{\alpha\beta}B^{ij}f_{,ij}\left(n-1\right)+C_{d}^{i}f_{,i}g^{cd}\left(\begin{array}{c} g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha} \\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab} \end{array}\right) \\ +C_{a}^{i}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C_{\alpha}^{i}f_{,i}g^{ab}g_{\beta b,a}-C_{\beta}^{i}f_{,i}g^{ab}g_{\alpha b,a} \end{array}\right). \tag{47}$$

3.1.1.2 The Metric Dirac Equation

Going back to (31), leaving F unfixed and reordering:

$$
\begin{gathered}
0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}\\
=\left(\begin{array}{c}
R_{\alpha\beta}-\dfrac{F'}{2F}\left(\begin{array}{c}
2g_{\alpha\beta}B^{ij}f_{,ij}\left(n-1\right)+C_{d}^{i}f_{,i}g^{cd}\left(\begin{array}{c}
g_{\alpha c,\beta}-\dfrac{1}{2}ng_{\alpha c,\beta}-\dfrac{1}{2}ng_{\beta c,\alpha}\\
+\dfrac{1}{2}ng_{\alpha\beta,c}+\dfrac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}
\end{array}\right)\\
+C_{a}^{i}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C_{\alpha}^{i}f_{,i}g^{ab}g_{\beta b,a}-C_{\beta}^{i}f_{,i}g^{ab}g_{\alpha b,a}
\end{array}\right)\\
-\left(R-\dfrac{F'}{2F}\left(\begin{array}{c}
\left(n-1\right)\left(2n\cdot B^{ij}f_{,ij}+C_{d}^{i}f_{,i}g^{cd}g^{ab}g_{ab,c}\right)\\
-nC_{d}^{i}f_{,i}g^{cd}g^{ab}g_{ac,b}
\end{array}\right)\right)\cdot\left(\dfrac{1}{2}+H\right)g_{\alpha\beta}\\
-\left(n-1\right)\dfrac{g_{\alpha\beta}}{4F^{2}}C^{ij}f_{,i}f_{,j}\left(4FF''+\left(F'\right)^{2}\left(n-6\right)\right)\left(1-n\cdot\left(\dfrac{1}{2}+H\right)\right)
\end{array}\right)\delta G^{\alpha\beta}
\end{gathered}
,\qquad(48)
$$

now gives us the opportunity to derive a quantum gravity Dirac equation. Knowing that, instead of condition (5), respectively, (7), (8), and (9), we could also have any other condition in (48) like e.g., (with an arbitrary function $H'[f]=\frac{\partial H[f]}{\partial f}=H'$):

$$
\begin{gathered}
\frac{4FF''+\left(F'\right)^{2}\left(n-6\right)}{4F}=F'\cdot H'\\
\Rightarrow\quad F=\begin{cases}
\left(C_{f0}+\displaystyle\int_{1}^{f}e^{H[\phi]}\cdot\frac{n-2}{4}\cdot d\phi\right)^{\frac{4}{n-2}}\cdot C_{f1} & n>2\\
C_{f1}\cdot e^{C_{f0}\cdot\int_{1}^{f}e^{H[\phi]}d\phi} & n=2
\end{cases}
\end{gathered}
,\qquad(49)
$$

leading us to:

$$
\begin{gathered}
0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}\\
=\left(\begin{array}{c}
R_{\alpha\beta}-R\cdot\left(\dfrac{1}{2}+H\right)g_{\alpha\beta}\\
-\dfrac{F'}{2F}\left(\begin{array}{c}
2g_{\alpha\beta}B^{ij}f_{,ij}\left(n-1\right)+C_{d}^{i}f_{,i}g^{cd}\left(\begin{array}{c}
g_{\alpha c,\beta}-\dfrac{1}{2}ng_{\alpha c,\beta}-\dfrac{1}{2}ng_{\beta c,\alpha}\\
+\dfrac{1}{2}ng_{\alpha\beta,c}+\dfrac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}
\end{array}\right)\\
+C_{a}^{i}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C_{\alpha}^{i}f_{,i}g^{ab}g_{\beta b,a}-C_{\beta}^{i}f_{,i}g^{ab}g_{\alpha b,a}
\end{array}\right)\\
+\left(\dfrac{F'}{2F}\left(\begin{array}{c}
\left(n-1\right)\left(2n\cdot B^{ij}f_{,ij}+C_{d}^{i}f_{,i}g^{cd}g^{ab}g_{ab,c}\right)\\
-nC_{d}^{i}f_{,i}g^{cd}g^{ab}g_{ac,b}
\end{array}\right)\right)\cdot\left(\dfrac{1}{2}+H\right)g_{\alpha\beta}\\
-\left(n-1\right)\dfrac{g_{\alpha\beta}}{F}C^{ij}f_{,i}f_{,j}F'\cdot H'\left(1-n\cdot\left(\dfrac{1}{2}+H\right)\right)
\end{array}\right)\delta G^{\alpha\beta}
\end{gathered}
,\qquad(50)
$$

where we have separated what—we think—classically (with H=0) is the gravity part. We can reshape (50) as follows:

$$
\begin{aligned}
0 &= \left(R^{*}_{\alpha\beta} - R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta} + g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}} \\
&= \left(\begin{array}{c}
\frac{2F}{F'}\left(R_{\alpha\beta} - R\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\right) \\
-C^{i}_{d}f_{,i}g^{cd}\left(g_{\alpha c,\beta} - \frac{1}{2}ng_{\alpha c,\beta} - \frac{1}{2}ng_{\beta c,\alpha} + \frac{1}{2}ng_{\alpha\beta,c}\right) \\
-\left(C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha} - g_{\beta\alpha,b}\right) - C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a} - C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a}\right) \\
+g_{\alpha\beta}\left(\begin{array}{c}
\left((n-1)\left(2n\cdot B^{ij}f_{,ij} + C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right) - nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b}\right)\cdot\left(\frac{1}{2}+H\right) \\
-\left(2B^{ij}f_{,ij}(n-1) + C^{i}_{d}f_{,i}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right) \\
-2(n-1)C^{ij}f_{,i}f_{,j}\cdot H'\left(1-n\cdot\left(\frac{1}{2}+H\right)\right)
\end{array}\right)
\end{array}\right)\delta G^{\alpha\beta}
\end{aligned}
\quad . \qquad (51)
$$

We have a variety of options to solve this equation. Here we are only interested in extracting the classical Dirac equation (though more general than the classical one as we keep both the coordinates and the number of dimensions arbitrary).

3.1.1.2.1 Coupling Gravity and Quantum Effects

Thus, we start with the following separation:

$$
\begin{aligned}
0 &= \left(\begin{array}{c}
\frac{2F}{F'}\left(R_{\alpha\beta} - R\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\right) \\
-C^{i}_{d}f_{,i}g^{cd}\left(g_{\alpha c,\beta} - \frac{1}{2}ng_{\alpha c,\beta} - \frac{1}{2}ng_{\beta c,\alpha} + \frac{1}{2}ng_{\alpha\beta,c}\right) \\
-\left(C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha} - g_{\beta\alpha,b}\right) - C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a} - C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a}\right)
\end{array}\right) \\
0 &= \left(\begin{array}{c}
\left((n-1)\left(2n\cdot B^{ij}f_{,ij} + C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right) - nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b}\right)\cdot\left(\frac{1}{2}+H\right) \\
-\left(2B^{ij}f_{,ij}(n-1) + C^{i}_{d}f_{,i}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right) \\
-2(n-1)C^{ij}f_{,i}f_{,j}\cdot H'\left(1-n\cdot\left(\frac{1}{2}+H\right)\right)
\end{array}\right)
\end{aligned}
\quad . \qquad (52)
$$

Now we demand an eigenvalue solution for the following term:

$$
-M^{2}f = \left(\begin{array}{c}
\left((n-1)\left(2n\cdot B^{ij}f_{,ij} + C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right) - nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b}\right)\cdot\left(\frac{1}{2}+H\right) \\
-\left(2B^{ij}f_{,ij}(n-1) + C^{i}_{d}f_{,i}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right)
\end{array}\right), \qquad (53)
$$

leading us to:

$$0=M^2 f+2(n-1)C^{ij}f_{,i}f_{,j}\cdot H'\left(1-n\cdot\left(\frac{1}{2}+H\right)\right). \tag{54}$$

The simplest path forward in order to obtain a good starting point for the Dirac equation is to apply a function H'=H'[f]=Y/f. The corresponding solution for F[f] in the case of n>2 would be:

$$\frac{4FF''+(F')^2(n-6)}{4F}=F'\cdot H'=Y\cdot\frac{F'}{f}$$
$$\Rightarrow\quad\left\{\begin{array}{ll} F=C_{f1}\cdot\left(C_{f0}+f^{1+Y}\right)^{\frac{4}{n-2}} & Y\neq-1, n>2\\ F=C_{f1}\cdot\left(C_{f0}+\ln[f]\right)^{\frac{4}{n-2}} & Y=-1, n>2\end{array}\right\}. \tag{55}$$

By introducing the index i for a list of functions f_i, (54) becomes[1]:

$$0=M^2 f_i^2+2(n-1)C^{\alpha\beta}f_{i,\alpha}f_{i,\beta}\cdot\left(1-n\cdot\left(\frac{1}{2}+H\right)\right)$$
$$\xrightarrow{m^2=\frac{M^2}{2(n-1)\cdot\left(1-n\cdot\left(\frac{1}{2}+H\right)\right)}}$$
$$0=m^2 f_i^2+C^{\alpha\beta}f_{i,\alpha}f_{i,\beta}. \tag{56}$$

Thereby we have set Y=1. The next step is to apply the known relation between the Dirac matrices, which we here give in the original Dirac assumption with n=4, R=0, and for Cartesian coordinates (note that all empty slots are standing for zeros; also note that we here only give the Dirac matrices in the Cartesian, respectively the Minkowski case for a four-dimensional space-time):

$$\gamma^0=\begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix};\quad\gamma^1=\begin{pmatrix}&&&1\\&&1&\\&-1&&\\-1&&&\end{pmatrix}$$
$$\gamma^2=\begin{pmatrix}&&&-i\\&&i&\\&i&&\\-i&&&\end{pmatrix};\quad\gamma^3=\begin{pmatrix}&&1&\\&&&-1\\-1&&&\\&1&&\end{pmatrix};\quad I=\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}. \tag{57}$$

We also introduce the following relation with our matrix object C^{ij} and the metric tensor, reading:

$$C^{ij}\to C^{\alpha\beta}=\frac{\gamma^\alpha\gamma^\beta+\gamma^\beta\gamma^\alpha}{2}, \tag{58}$$

and put (56) into the following form:

[1] It can be shown that such a list naturally and without any postulations arises either within an Everett "multiverse" approach (e.g., [10, 11]) or via the assumption of a multi-factor scaling of the metric tensor as we are going to consider in subsection "Towards the Dirac Spinors" and the section "The Bianchi Separator" in our book [5].

$$
\begin{aligned}
& f_{i,\alpha} f_{i,\beta}\left(\frac{\gamma^{\alpha}\gamma^{\beta}+\gamma^{\beta}\gamma^{\alpha}}{2}\right)+\frac{I+I}{2}\cdot m^{2}\cdot f_{i}^{2}=0 \\
& \Rightarrow f_{i,\alpha} f_{i,\beta}\gamma^{\alpha}\gamma^{\beta}+I\cdot m^{2}\cdot f_{i}^{2} \\
& =\left(f_{i,\alpha}\gamma^{\alpha} f_{i,\beta}\gamma^{\beta}+i\cdot I\cdot m\cdot f_{i}\overbrace{\left(f_{i,\beta}\gamma^{\beta}-f_{i,\alpha}\gamma^{\alpha}\right)}^{f_{i,\beta}\gamma^{\beta}-f_{i,\alpha}\gamma^{\alpha}=f_{i,\gamma}\gamma^{\gamma}-f_{i,\gamma}\gamma^{\gamma}=0}+I\cdot m^{2}\cdot f_{i}^{2}\right) \\
& =\left(f_{i,\alpha}\gamma^{\alpha}+i\cdot I\cdot m\cdot f_{i}\right)\left(f_{i,\beta}\gamma^{\beta}-i\cdot I\cdot m\cdot f_{i}\right)=0 \\
& \left(f_{i,\beta}\gamma^{\beta}-i\cdot I\cdot m\cdot f_{i}\right)=0 \\
& \Rightarrow f_{i,\alpha} f_{i,\beta}\gamma^{\beta}\gamma^{\alpha}+I\cdot m^{2}\cdot f_{i}^{2}=0 \\
& \Rightarrow\left(f_{i,\alpha}\gamma^{\alpha}+i\cdot I\cdot m\cdot f_{i}\right)\left(f_{i,\beta}\gamma^{\beta}-i\cdot I\cdot m\cdot f_{i}\right)=0
\end{aligned}
\quad . \tag{59}
$$

In the last line we recognize the Dirac equations:

$$f_{i,\alpha}\gamma^{\alpha}\pm i\cdot I\cdot m\cdot f_{i}=0\,. \tag{60}$$

Unfortunately, in the interesting cases for n>2 and *H*=0 our equation (56) changes to:

$$
\begin{aligned}
& 0=M^{2}f_{i}^{2}-(n-1)\cdot C^{\alpha\beta}f_{i,\alpha}f_{i,\beta}\cdot(n-2) \\
& \xrightarrow{m^{2}=\frac{M^{2}}{(n-1)\cdot(n-2)}} \\
& 0=m^{2}f_{i}^{2}-C^{\alpha\beta}f_{i,\alpha}f_{i,\beta}
\end{aligned}
\quad . \tag{61}
$$

In order to end up with the desired "+"-sign, which is to say with:

$$0=m^{2}f_{i}^{2}+C^{\alpha\beta}f_{i,\alpha}f_{i,\beta} \tag{62}$$

again, we would need to demand Y=-1 in (55).

More discussion on the connection with the classical case, including the full derivation with respect to the eigenvalue equation of Klein-Gordon character in (53), is been given in [2] in the section "How to Metrically Derive the Dirac Equation".

3.1.1.2.2 Decoupling Gravity and Quantum Effects

As said before, there are many options to combine the various terms in the main equation (51). An interesting one is the complete separation of Einstein's gravity in the vacuum case and the quantum effects. We therefore demand:

$$
\begin{gathered}
0=\frac{2F}{F'}\left(R_{\alpha\beta}-R\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\right)\\
\Rightarrow 0=R_{\alpha\beta}-R\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}
\end{gathered}
$$

$$
0=\begin{pmatrix}
+g_{\alpha\beta}\begin{pmatrix}
\left((n-1)\left(2n\cdot B^{ij}f_{,ij}+C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right)-nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b}\right)\cdot\left(\frac{1}{2}+H\right)\\
-\left(2B^{ij}f_{,ij}(n-1)+C^{i}_{d}f_{,i}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right)\\
-2(n-1)C^{ij}f_{,i}f_{,j}\cdot H'\left(1-n\cdot\left(\frac{1}{2}+H\right)\right)
\end{pmatrix}\\
-C^{i}_{d}f_{,i}g^{cd}\left(g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}+\frac{1}{2}ng_{\alpha\beta,c}\right)\\
-\left(C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a}-C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a}\right)
\end{pmatrix}. \tag{63}
$$

Now we demand an eigenvalue solution for the following term:

$$
-g_{\alpha\beta}M^{2}f=\begin{pmatrix}
+g_{\alpha\beta}\begin{pmatrix}
\begin{pmatrix}
(n-1)\left(2n\cdot B^{ij}f_{,ij}+C^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ab,c}\right)\\
-nC^{i}_{d}f_{,i}g^{cd}g^{ab}g_{ac,b}
\end{pmatrix}\cdot\left(\frac{1}{2}+H\right)\\
-\left(2B^{ij}f_{,ij}(n-1)+C^{i}_{d}f_{,i}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right)
\end{pmatrix}\\
-C^{i}_{d}f_{,i}g^{cd}\left(g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}+\frac{1}{2}ng_{\alpha\beta,c}\right)\\
-\left(C^{i}_{a}f_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C^{i}_{\alpha}f_{,i}g^{ab}g_{\beta b,a}-C^{i}_{\beta}f_{,i}g^{ab}g_{\alpha b,a}\right)
\end{pmatrix}, \tag{64}
$$

leading us, as before, to:

$$
\begin{gathered}
0=g_{\alpha\beta}M^{2}f+g_{\alpha\beta}2(n-1)C^{ij}f_{,i}f_{,j}\cdot H'\left(1-n\cdot\left(\frac{1}{2}+H\right)\right)\\
\Rightarrow\\
0=M^{2}f+2(n-1)C^{ij}f_{,i}f_{,j}\cdot H'\left(1-n\cdot\left(\frac{1}{2}+H\right)\right)
\end{gathered}. \tag{65}
$$

3.1.1.2.3 The Dirac-Matter Separation

Finally, we investigate a separation of Dirac-matter and the Einstein field equations with matter following from our quantum gravity equations (51) when splitting up and demanding an eigenvalue solution as follows:

$$-g_{\alpha\beta}M^2 f = \begin{pmatrix} \frac{2F}{F'}\left(R_{\alpha\beta} - R\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\right) \\ -C^i_d f_{,i} g^{cd}\left(g_{\alpha c,\beta} - \frac{1}{2} n g_{\alpha c,\beta} - \frac{1}{2} n g_{\beta c,\alpha} + \frac{1}{2} n g_{\alpha\beta,c}\right) \\ -\left(C^i_a f_{,i} g^{ab}\left(g_{\beta b,\alpha} - g_{\beta\alpha,b}\right) - C^i_\alpha f_{,i} g^{ab} g_{\beta b,a} - C^i_\beta f_{,i} g^{ab} g_{\alpha b,a}\right) \\ +g_{\alpha\beta}\begin{pmatrix} \begin{pmatrix} (n-1)\left(2n\cdot B^{ij}f_{,ij} + C^i_d f_{,i} g^{cd} g^{ab} g_{ab,c}\right) \\ -nC^i_d f_{,i} g^{cd} g^{ab} g_{ac,b}\end{pmatrix}\cdot\left(\frac{1}{2}+H\right) \\ -\left(2B^{ij}f_{,ij}(n-1) + C^i_d f_{,i} g^{cd}\frac{1}{2} g_{ab,c}g^{ab}\right)\end{pmatrix}\end{pmatrix}. \tag{66}$$

This time, where we only have two equations instead of three (!), we obtain the total mass for our Dirac equation derivation from (65) via (62) and (59) to (60) from both the quantum matter:

$$\begin{pmatrix} g_{\alpha\beta}\begin{pmatrix} \left((n-1)\left(2n\cdot B^{ij}f_{,ij} + C^i_d f_{,i} g^{cd} g^{ab} g_{ab,c}\right) - nC^i_d f_{,i} g^{cd} g^{ab} g_{ac,b}\right)\cdot\left(\frac{1}{2}+H\right) \\ -\left(2B^{ij}f_{,ij}(n-1) + C^i_d f_{,i} g^{cd}\frac{1}{2} g_{ab,c}g^{ab}\right)\end{pmatrix} \\ -C^i_d f_{,i} g^{cd}\left(g_{\alpha c,\beta} - \frac{1}{2} n g_{\alpha c,\beta} - \frac{1}{2} n g_{\beta c,\alpha} + \frac{1}{2} n g_{\alpha\beta,c}\right) \\ -\left(C^i_a f_{,i} g^{ab}\left(g_{\beta b,\alpha} - g_{\beta\alpha,b}\right) - C^i_\alpha f_{,i} g^{ab} g_{\beta b,a} - C^i_\beta f_{,i} g^{ab} g_{\alpha b,a}\right)\end{pmatrix} \tag{67}$$

and the curvature terms:

$$\frac{2F}{F'}\left(R_{\alpha\beta} - R\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\right). \tag{68}$$

The Dirac equation we then obtain, thereby following the path of setting (66) into (51), resulting in (65) and performing the rest of the derivation as given above, would seal the quantum gravity calculation. In the case of metrics of constants, (67) and (68) simplify dramatically and make (66) to an almost Klein-Gordon-like classical equation:

$$-M^2 f = 2\cdot(n-1)B^{ij}f_{,ij}\cdot\left(n\cdot\left(\frac{1}{2}+H\right)-1\right). \tag{69}$$

One would only need to demand $B^{ij} = I\cdot g^{ij}$ to truly have a Klein-Gordon equation.

Now we do not only know where the Dirac equation and the Dirac spinors (see [5, 6], section "The Bianchi Separator") are coming from, but also how to derive a Dirac equation in a curved space-time and an arbitrary number of dimensions. We have also learned that the linearity of the Dirac equation requires a certain restriction with respect to the wave function solutions, namely (24).

3.1.1.2.4 Towards the Dirac Spinors

So far, the list of Dirac functions was classically postulated or derived as a by-product of the Everett multiverse theory (e.g., [2, 10]). Here now we intend to obtain this spinor by starting with a function vector within our metric derivation. At first, we introduce the following wave function:

$$\not{X} = \Phi_D \cdot \Psi^D = \sum_{i=1}^{N} \Phi_i \cdot \Psi^i \,, \tag{70}$$

within the wrapper F:

$$F = F_j = F_j\left[\not{X}\right] = F_j\left[\Phi_D \cdot \Psi^D\right] = F_j\left[\sum_{i=1}^{N} \Phi_i \cdot \Psi^i\right], \tag{71}$$

and would then obtain from (51):

$$\begin{array}{c}
0 = \left(R^{*}{}_{\alpha\beta} - R^{*}\left(\frac{1}{2}+H\right) G_{\alpha\beta} \right) \overbrace{\left(\frac{1}{F} \cdot \delta g^{\alpha\beta} + g^{\alpha\beta} \cdot \delta\left(\frac{1}{F}\right) \right)}^{\delta G^{\alpha\beta}} \\
= \left(\begin{array}{c}
\frac{2F}{F'}\left(R_{\alpha\beta} - R \cdot \left(\frac{1}{2}+H\right) g_{\alpha\beta} \right) \\
- C_d^i \not{X}_{,i} g^{cd} \left(g_{\alpha c,\beta} - \frac{1}{2} n g_{\alpha c,\beta} - \frac{1}{2} n g_{\beta c,\alpha} + \frac{1}{2} n g_{\alpha\beta,c} \right) \\
- \left(C_a^i \not{X}_{,i} g^{ab} \left(g_{\beta b,\alpha} - g_{\beta\alpha,b} \right) - C_\alpha^i \not{X}_{,i} g^{ab} g_{\beta b,a} - C_\beta^i \not{X}_{,i} g^{ab} g_{\alpha b,a} \right) \\
+ g_{\alpha\beta} \left(\begin{array}{c}
\left((n-1)\left(2n \cdot B^{ij} \not{X}_{,ij} + C_d^i \not{X}_{,i} g^{cd} g^{ab} g_{ab,c} \right) - n C_d^i \not{X}_{,i} g^{cd} g^{ab} g_{ac,b} \right) \cdot \left(\frac{1}{2}+H \right) \\
- \left(2B^{ij} \not{X}_{,ij} (n-1) + C_d^i \not{X}_{,i} g^{cd} \frac{1}{2} g_{ab,c} g^{ab} \right) \\
-2(n-1) C^{ij} \not{X}_{,i} \not{X}_{,j} \cdot H' \left(1 - n \cdot \left(\frac{1}{2}+H \right) \right)
\end{array} \right)
\end{array} \right) \delta G^{\alpha\beta}
\end{array} \tag{72}$$

with:

$$\begin{array}{c}
\not{X}_{,\alpha\beta} = \left(\Phi_D \cdot \Psi^D \right)_{,\alpha\beta} = \Phi_{D,\alpha\beta} \cdot \Psi^D + \Phi_{D,\alpha} \cdot \Psi^D{}_{,\beta} + \Phi_{D,\beta} \cdot \Psi^D{}_{,\alpha} + \Phi_D \cdot \Psi^D{}_{,\alpha\beta} \\
\not{X}_{,\alpha} = \left(\Phi_D \cdot \Psi^D \right)_{,\alpha} = \Phi_{D,\alpha} \cdot \Psi^D + \Phi_D \cdot \Psi^D{}_{,\alpha} \\
\not{X}_{,\alpha} \not{X}_{,\beta} = \left(\Phi_D \cdot \Psi^D \right)_{,\alpha} \left(\Phi_D \cdot \Psi^D \right)_{,\beta} = \left(\begin{array}{c}
\Phi_{D,\alpha} \cdot \Psi^D \cdot \Phi_{D,\beta} \cdot \Psi^D + \Phi_D \cdot \Psi^D{}_{,\alpha} \cdot \Phi_D \cdot \Psi^D{}_{,\beta} \\
+ \Phi_{D,\alpha} \cdot \Psi^D{}_{,\beta} \cdot \Phi_D \cdot \Psi^D + \Phi_{D,\beta} \cdot \Psi^D{}_{,\alpha} \cdot \Phi_D \cdot \Psi^D
\end{array} \right)
\end{array} . \tag{73}$$

This simplifies significantly when we assume one of the “vectors” to be a list (or whatever) of constants, because then we just have:

$$\begin{array}{c}
\not{X}_{,\alpha\beta} = \left(\Phi_D \cdot \Psi^D \right)_{,\alpha\beta} = \Phi_{D,\alpha\beta} \cdot \Psi^D \\
\not{X}_{,\alpha} = \left(\Phi_D \cdot \Psi^D \right)_{,\alpha} = \Phi_{D,\alpha} \cdot \Psi^D \\
\not{X}_{,\alpha} \not{X}_{,\beta} = \left(\Phi_D \cdot \Psi^D \right)_{,\alpha} \left(\Phi_D \cdot \Psi^D \right)_{,\beta} = \Phi_{D,\alpha} \cdot \Phi_{D,\beta} \cdot \Psi^D \cdot \Psi^D
\end{array} . \tag{74}$$

As we have learned in [5, 6], section “The Bianchi Separator”, this gives us a great variety of options to produce linear field equations by applying the technique introduced and explained above.

Here we just follow the path of the previous subsection, which is to say “The Dirac-Matter Separation”. Following all the steps there is easy because we only have to substitute the ordinary wave function f by our new scalar product one $\not{X}$. This gives us—instead of (65)—the following result:

$$0=g_{\alpha\beta}\frac{F'}{F}M^2\Psi+\frac{g_{\alpha\beta}}{4F^2}C^{ij}\Psi_{,i}\Psi_{,j}\left(n-1\right)\left(4FF''+(F')^2\left(n-6\right)\right)\left(\frac{1}{2}-H\right) \tag{75}$$

plus the corresponding eigen equation:

$$\begin{array}{c} -g_{\alpha\beta}M^2\Psi \\ =\left(\begin{array}{c} \frac{2F}{F'}\left(R_{\alpha\beta}-R\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\right) \\ -C_d^i\Psi_{,i}g^{cd}\left(g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}+\frac{1}{2}ng_{\alpha\beta,c}\right) \\ -\left(C_a^i\Psi_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C_\alpha^i\Psi_{,i}g^{ab}g_{\beta b,a}-C_\beta^i\Psi_{,i}g^{ab}g_{\alpha b,a}\right) \\ +g_{\alpha\beta}\left(\begin{array}{c}\left((n-1)\left(2n\cdot B^{ij}\Psi_{,ij}+C_d^i\Psi_{,i}g^{cd}g^{ab}g_{ab,c}\right)-nC_d^i\Psi_{,i}g^{cd}g^{ab}g_{ac,b}\right)\cdot\left(\frac{1}{2}+H\right) \\ -\left(2B^{ij}\Psi_{,ij}\left(n-1\right)+C_d^i\Psi_{,i}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right)\end{array}\right)\end{array}\right).\end{array} \tag{76}$$

As in the subsection above ("The Dirac-Matter Separation"), we only have two equations and we obtain the total mass for our Dirac equation from (76) containing both the quantum matter:

$$\left(\begin{array}{c} -C_d^i\Psi_{,i}g^{cd}\left(g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}+\frac{1}{2}ng_{\alpha\beta,c}\right) \\ -\left(C_a^i\Psi_{,i}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-C_\alpha^i\Psi_{,i}g^{ab}g_{\beta b,a}-C_\beta^i\Psi_{,i}g^{ab}g_{\alpha b,a}\right) \\ +g_{\alpha\beta}\left(\begin{array}{c}\left((n-1)\left(2n\cdot B^{ij}\Psi_{,ij}+C_d^i\Psi_{,i}g^{cd}g^{ab}g_{ab,c}\right)-nC_d^i\Psi_{,i}g^{cd}g^{ab}g_{ac,b}\right)\cdot\left(\frac{1}{2}+H\right) \\ -\left(2B^{ij}\Psi_{,ij}\left(n-1\right)+C_d^i\Psi_{,i}g^{cd}\frac{1}{2}g_{ab,c}g^{ab}\right)\end{array}\right)\end{array}\right) \tag{77}$$

and the curvature terms:

$$\frac{2F}{F'}\left(R_{\alpha\beta}-R\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\right). \tag{78}$$

The Dirac equation then seals the quantum gravity calculation. Thereby we have to remember that—this time—our wave function is a scalar product and when choosing H' accordingly (see above), we obtain:

$$\begin{array}{c} 0=g_{\alpha\beta}\frac{F'}{F}M^2\Psi+\frac{g_{\alpha\beta}}{4F^2}C^{ij}\Psi_{,i}\Psi_{,j}\left(n-1\right)H'\left(\frac{1}{2}-H\right) \\ =g_{\alpha\beta}M^2\Psi^2+g_{\alpha\beta}C^{ij}\Psi_{,i}\Psi_{,j} \\ \Rightarrow 0=M^2\Psi^2+C^{ij}\Psi_{,i}\Psi_{,j}\end{array}. \tag{79}$$

Factorization yields:

$$0=\left(M\cdot\Psi+i\cdot C^i\Psi_{,i}\right)\left(M\cdot\Psi-i\cdot C^j\Psi_{,j}\right). \tag{80}$$

Now we incorporate (70) and obtain:

$$0=\left(M\cdot\Phi_D\cdot\Psi^D+i\cdot C^i\Phi_{D,i}\cdot\Psi^D\right)\left(M\cdot\Phi_D\cdot\Psi^D-i\cdot C^j\Phi_{D,j}\cdot\Psi^D\right). \tag{81}$$

From there we can just factor out the list of constants Ψ^D. We realize that the list of functions in the original Dirac approach comes from a quantum gravity evaluation originating from the Einstein-Hilbert action with a scaled metric tensor. The scalar factor itself is a scalar product and explains the occurrence of the Dirac spinors in a natural way. We obtain the classical Dirac equations as follows:

$$\begin{aligned}0&=\left(M\cdot\Phi_D+i\cdot C^i\Phi_{D,i}\right)\cdot\Psi^D\left(M\cdot\Phi_D-i\cdot C^j\Phi_{D,j}\right)\cdot\Psi^D\\ \Rightarrow 0&=M\cdot\Phi_D\pm i\cdot C^i\Phi_{D,i}\end{aligned}, \tag{82}$$

which are quantum gravity Dirac equations when taking into account that the term for M comes as eigenvalue from the field equations (76). We simply substitute the—so far—unfixed objects C^i by the Dirac matrices and would have finished the derivation of the classical Dirac equation from a holistic quantum gravity approach (without incorporating any approximations):

$$0=M\cdot\Phi_D\pm i\cdot\gamma^i\Phi_{D,i}. \tag{83}$$

We have generalized our approach elsewhere [5, 6] and seen that many Dirac spinors can be constructed at various scaling levels of scaled metrics (see [5, 6], section "The Bianchi Separator").

4 Some More Rigorous Considerations of the "Transformers Linearity"

Having seen in the section above how a Dirac theory can be extracted from the quantum gravity field equations of a scaled metric tensor, we now want to investigate this transformation technique a bit more rigorously. Our intention is to get rid of the nonlinearity with respect to the metric volume function f when introducing a scaled metric of the type (1) or—more complex and flexible:

$$\begin{aligned}\boxed{G_A}_{\alpha\beta}&=g_{\alpha\beta}\cdot\prod_{i=1}^{A}F_i\\ \boxed{G_0}_{\alpha\beta}&=g_{\alpha\beta}\end{aligned}. \tag{84}$$

Concentrating on the simpler form (1) and the generalized Hamilton principle (2), we result in the quantum gravity field equations (3), which we would like to reshape a bit as follows:

$$0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\overbrace{\left(\frac{1}{F}\cdot\delta g^{\alpha\beta}+g^{\alpha\beta}\cdot\delta\left(\frac{1}{F}\right)\right)}^{\delta G^{\alpha\beta}}$$

$$=\left(\begin{array}{c}\left(R_{\alpha\beta}-\frac{F'}{2F}\left(\begin{array}{c}f_{,\alpha\beta}(n-2)+f_{,ab}g_{\alpha\beta}g^{ab}+f_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-f_{,\alpha}g^{ab}g_{\beta b,a}\\ -f_{,\beta}g^{ab}g_{\alpha b,a}+f_{,d}g^{cd}\left(\begin{array}{c}g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}\\ +\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\end{array}\right)\end{array}\right)\right)\\ -\left(R-\frac{F'}{2F}\left(\begin{array}{c}(n-1)\left(2g^{ab}f_{,ab}+f_{,d}g^{cd}g^{ab}g_{ab,c}\right)\\ -nf_{,d}g^{cd}g^{ab}g_{ac,b}\end{array}\right)\right)\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\\ +\frac{1}{4F^{2}}\left(\begin{array}{c}f_{,\alpha}\cdot f_{,\beta}(n-2)\left(3(F')^{2}-2FF''\right)\\ +g_{\alpha\beta}f_{,c}f_{,d}g^{cd}\left(\begin{array}{c}\left((F')^{2}(4-n)-2FF''\right)\\ +(n-1)\cdot\left(\frac{1}{2}+H\right)\left(4FF''+(F')^{2}(n-6)\right)\end{array}\right)\end{array}\right)\end{array}\right)\delta G^{\alpha\beta}\quad. \qquad (85)$$

We realize the term $f_{,\alpha}\cdot f_{,\beta}$ as the one hindering us to obtain just one equation for the wrapper function F[f] in order to make the nonlinear differential terms in f to disappear. Hence, we are looking for a way to transform this factor into something which would mirror its scalar partner $f_{,c}f_{,d}g^{cd}$. Starting with the ansatz for the derivative of f as follows:

$$\begin{array}{c}f_{,\alpha}\cdot ?^{k}=\gamma_{\alpha}^{ik}\varphi_{,i}\\ f_{,\alpha}\cdot ?_{k}=\gamma_{k\alpha}^{i}\varphi_{,i}\end{array}, \qquad (86)$$

we would obtain the following expression for our "problematic" factor $f_{,\alpha}\cdot f_{,\beta}$:

$$f_{,\alpha}\cdot f_{,\beta}=f_{,\alpha}\cdot ?^{k}\cdot f_{,\beta}\cdot ?_{k}=\frac{\gamma_{\alpha}^{ik}\gamma_{k\beta}^{j}+\gamma_{k\beta}^{j}\gamma_{\alpha}^{ik}}{2}\varphi_{,i}\varphi_{,j}=g_{\alpha\beta}\cdot C^{ij}\varphi_{,i}\varphi_{,j}, \qquad (87)$$

where we assumed to apply something similar to the Dirac matrices, because we know that these satisfy the conditions:

$$g^{\alpha\beta}\cdot I=\frac{\gamma^{\alpha}\gamma^{\beta}+\gamma^{\beta}\gamma^{\alpha}}{2};\quad g_{\alpha\beta}\cdot I=\frac{\gamma_{\alpha}\gamma_{\beta}+\gamma_{\beta}\gamma_{\alpha}}{2}. \qquad (88)$$

Thereby the object I is known to be the unit matrix and hence, we better write:

$$g^{\alpha\beta}\cdot I^{ij}=\frac{\gamma^{\alpha}\gamma^{\beta}+\gamma^{\beta}\gamma^{\alpha}}{2};\quad g_{\alpha\beta}\cdot I^{ij}=\frac{\gamma_{\alpha}\gamma_{\beta}+\gamma_{\beta}\gamma_{\alpha}}{2}. \qquad (89)$$

The objects in (87), however, should—at the moment—be understood more generally and undetermined. Consequently, we may also demand the following:

$$f_{,\alpha}\cdot g_{ab,\beta}=f_{,\alpha}\cdot ?^{k}\cdot g_{ab,\beta}\cdot ?_{k}=\frac{\gamma_{\alpha}^{ik}\gamma_{k\beta}^{j}+\gamma_{k\beta}^{j}\gamma_{\alpha}^{ik}}{2}\varphi_{,i}\Gamma_{ab,j}=g_{\alpha\beta}\cdot C^{ij}\varphi_{,i}\Gamma_{ab,j}, \qquad (90)$$

and:

$$\begin{aligned} f_{,\alpha\beta} \rightarrow f_{,\alpha\beta}\cdot ?^{k}\cdot ?_{k} \rightarrow \left(f_{,\alpha}\cdot ?^{k}\right)_{,\beta}\cdot ?_{k} &= \frac{\left(\gamma_{\alpha}^{ik}\varphi_{,i}\right)_{,\beta}\cdot ?_{k}+\left(\gamma_{\beta}^{ik}\varphi_{,i}\right)_{,\alpha}\cdot ?_{k}}{2} \\ =\frac{\gamma_{k\beta}^{j}\left(\gamma_{\alpha}^{ik}\varphi_{,i}\right)_{,j}+\gamma_{k\alpha}^{j}\left(\gamma_{\beta}^{ik}\varphi_{,i}\right)_{,j}}{2} &= \frac{\gamma_{k\beta}^{j}\left(\gamma_{\alpha,j}^{ik}\varphi_{,i}+\gamma_{\alpha}^{ik}\varphi_{,ij}\right)+\gamma_{k\alpha}^{j}\left(\gamma_{\beta,j}^{ik}\varphi_{,i}+\gamma_{\beta}^{ik}\varphi_{,ij}\right)}{2} \\ &= \frac{\gamma_{k\beta}^{j}\gamma_{\alpha,j}^{ik}+\gamma_{k\alpha}^{j}\gamma_{\beta,j}^{ik}}{2}\varphi_{,i}+\frac{\gamma_{k\beta}^{j}\gamma_{\alpha}^{ik}+\gamma_{k\alpha}^{j}\gamma_{\beta}^{ik}}{2}\varphi_{,ij} \\ &= \frac{\gamma_{k\beta}^{j}\gamma_{\alpha,j}^{ik}+\gamma_{k\alpha}^{j}\gamma_{\beta,j}^{ik}}{2}\varphi_{,i}+g_{\alpha\beta}\cdot C^{ij}\varphi_{,ij} \end{aligned} . \qquad (91)$$

Note that the auxiliary metric Γ_{ab} should not be mistaken for the affine connection Γ_{ab}^{c}, which should not be a problem as the latter always has three indices and not two as the metric has. Now we assume that we are able to find objects (matrices, metrics, and wave functions) allowing us to even demand:

$$f_{,\alpha}\cdot f_{,\beta}=f_{,\alpha}\cdot ?^{k}\cdot f_{,\beta}\cdot ?_{k}=\frac{\gamma_{\alpha}^{ik}\gamma_{k\beta}^{j}+\gamma_{k\beta}^{j}\gamma_{\alpha}^{ik}}{2}f_{,i}f_{,j}=g_{\alpha\beta}\cdot C^{ij}f_{,i}f_{,j}, \qquad (92)$$

$$f_{,\alpha}\cdot g_{ab,\beta}=f_{,\alpha}\cdot ?^{k}\cdot g_{ab,\beta}\cdot ?_{k}=\frac{\gamma_{\alpha}^{ik}\gamma_{k\beta}^{j}+\gamma_{k\beta}^{j}\gamma_{\alpha}^{ik}}{2}f_{,i}g_{ab,j}=g_{\alpha\beta}\cdot C^{ij}f_{,i}g_{ab,j}, \qquad (93)$$

and:

$$\begin{aligned} f_{,\alpha\beta} \rightarrow f_{,\alpha\beta}\cdot ?^{k}\cdot ?_{k} \rightarrow \left(f_{,\alpha}\cdot ?^{k}\right)_{,\beta}\cdot ?_{k} &= \frac{\left(\gamma_{\alpha}^{ik}f_{,i}\right)_{,\beta}\cdot ?_{k}+\left(\gamma_{\beta}^{ik}f_{,i}\right)_{,\alpha}\cdot ?_{k}}{2} \\ =\frac{\gamma_{k\beta}^{j}\left(\gamma_{\alpha}^{ik}f_{,i}\right)_{,j}+\gamma_{k\alpha}^{j}\left(\gamma_{\beta}^{ik}f_{,i}\right)_{,j}}{2} &= \frac{\gamma_{k\beta}^{j}\left(\gamma_{\alpha,j}^{ik}f_{,i}+\gamma_{\alpha}^{ik}f_{,ij}\right)+\gamma_{k\alpha}^{j}\left(\gamma_{\beta,j}^{ik}f_{,i}+\gamma_{\beta}^{ik}f_{,ij}\right)}{2} \\ &= \frac{\gamma_{k\beta}^{j}\gamma_{\alpha,j}^{ik}+\gamma_{k\alpha}^{j}\gamma_{\beta,j}^{ik}}{2}f_{,i}+\frac{\gamma_{k\beta}^{j}\gamma_{\alpha}^{ik}+\gamma_{k\alpha}^{j}\gamma_{\beta}^{ik}}{2}f_{,ij} \\ &= \frac{\gamma_{k\beta}^{j}\gamma_{\alpha,j}^{ik}+\gamma_{k\alpha}^{j}\gamma_{\beta,j}^{ik}}{2}f_{,i}+g_{\alpha\beta}\cdot C^{ij}f_{,ij} \end{aligned} . \qquad (94)$$

We see that—so far—no function vector / list of functions / spinor is of need, but when we intend to separate (factorize), for instance (92), we detect some inconsistencies. We assume f to be a list of functions and start with the following trial for a split-up:

$$\begin{aligned} f^{D}{}_{,\alpha}\cdot f_{A,\beta} &= f^{D}{}_{,\alpha}\cdot ?^{k}\cdot f_{A,\beta}\cdot ?_{k}=\frac{\gamma_{\alpha}^{ik}\gamma_{k\beta}^{j}+\gamma_{k\beta}^{j}\gamma_{\alpha}^{ik}}{2}f^{D}{}_{,i}f_{A,j}=g_{\alpha\beta}\cdot C_{D}^{Aij}f^{D}{}_{,i}f_{A,j} \\ &= g_{\alpha\beta}\cdot B_{D}^{ij}B^{Aij}f^{D}{}_{,i}f_{A,j}=\frac{\gamma_{\alpha}^{iA}\gamma_{D\beta}^{j}+\gamma_{D\beta}^{j}\gamma_{\alpha}^{iA}}{2}f^{D}{}_{,i}f_{A,j}=g_{\alpha\beta}E_{A}^{j}E^{Di}f^{A}{}_{,j}f_{D,i} \end{aligned} . \qquad (95)$$

We realize that the rank of the product $E_{A}^{j}E^{Di}f^{A}{}_{,j}f_{D,i}$ on the right-hand side in the last line must be 2 and hence, some kind of matrix. Thus, we better write (?):

$$\begin{aligned} &\mathbf{f}_{,\alpha}\cdot\mathbf{f}_{,\beta}=g_{\alpha\beta}\mathbf{E}_{A}^{j}\mathbf{E}^{Di}\mathbf{f}^{A}{}_{,j}\mathbf{f}_{D,i} \\ &\Rightarrow f_{A\,,\alpha}^{D}\cdot f_{E\,,\beta}^{C}=g_{\alpha\beta}E_{A}^{Bj}f_{B\,,j}^{D}E_{E}^{Fi}f_{F\,,i}^{C} \end{aligned} . \qquad (96)$$

Of course, this could be made a scalar (thereby not counting the derivatives, of course) again via:

$$\Rightarrow f_{,\alpha}\cdot f_{,\beta}=f^{D}_{A,\alpha}\cdot f^{C}_{D,\beta}=g_{\alpha\beta}E^{Bj}_{A}f^{D}_{B,j}E^{Fi}_{D}f^{C}_{F,i}\Rightarrow f^{D}_{A,\alpha}\cdot f^{A}_{D,\beta}=g_{\alpha\beta}E^{Bj}_{A}f^{D}_{B,j}E^{Fi}_{D}f^{A}_{F,i}. \tag{97}$$

As an “observer” would see the latter terms just as what it—apparently—is, namely, a product of two gradients from a scalar:

$$f^{D}_{A,\alpha}\cdot f^{A}_{D,\beta}=g_{\alpha\beta}E^{Bj}_{A}f^{D}_{B,j}E^{Fi}_{D}f^{A}_{F,i}= \quad\Rightarrow\quad =f_{,\alpha}\cdot f_{,\beta}, \tag{98}$$

we can handle it as a scalar as long as we do not want to split the product up. This leads to the Klein-Gordon equivalent of our quantum gravity field equations. Thereby, when consequently performing the substitution in (85), without considering the split-up option at this stage, yields:

$$0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)$$

$$=\left(\begin{array}{c}
\left(R_{\alpha\beta}-\frac{F'}{2F}\left(\begin{array}{c}
\left(\frac{\gamma^{j}_{k\beta}\gamma^{ik}_{\alpha,j}+\gamma^{j}_{k\alpha}\gamma^{ik}_{\beta,j}}{2}f_{,i}+g_{\alpha\beta}\cdot C^{ij}f_{,ij}\right)(n-2)\\
+g_{\alpha\beta}\left(\frac{\gamma^{j}_{kb}\gamma^{ik}_{a,j}+\gamma^{j}_{ka}\gamma^{ik}_{b,j}}{2}g^{ab}f_{,i}+n\cdot C^{ij}f_{,ij}\right)\\
+g^{ab}C^{ij}f_{,i}\left(g_{a\alpha}g_{\beta b,j}-g_{ab}g_{\beta\alpha,j}\right)\\
-g^{ab}C^{ij}f_{,i}\left(g_{\alpha a}g_{\beta b,j}+g_{\beta a}g_{\alpha b,j}\right)\\
+g^{cd}C^{ij}f_{,i}\left(\begin{array}{c}
g_{d\beta}g_{\alpha c,j}-\frac{g_{d\beta}}{2}ng_{\alpha c,j}-\frac{g_{d\alpha}}{2}ng_{\beta c,j}\\
+\frac{g_{dc}}{2}ng_{\alpha\beta,j}+\frac{g_{dc}}{2}g_{\alpha\beta}g_{ab,j}g^{ab}
\end{array}\right)
\end{array}\right)\right)\\
-\left(R-\frac{F'}{2F}\left((n-1)\left(\begin{array}{c}
2\left(\frac{\gamma^{j}_{kb}\gamma^{ik}_{a,j}+\gamma^{j}_{ka}\gamma^{ik}_{b,j}}{2}g^{ab}f_{,i}+n\cdot C^{ij}f_{,ij}\right)\\
+C^{ij}f_{,i}g^{cd}g_{cd}g^{ab}g_{ab,j}\\
-nC^{ij}f_{,i}g^{cd}g_{bd}g^{ab}g_{ac,j}
\end{array}\right)\right)\right)\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\\
+\frac{g_{\alpha\beta}}{4F^{2}}\left(\begin{array}{c}
C^{ij}f_{,i}f_{,j}(n-2)\left(3(F')^{2}-2FF''\right)\\
+n\cdot C^{ij}f_{,i}f_{,j}\left(\begin{array}{c}
\left((F')^{2}(4-n)-2FF''\right)\\
+(n-1)\cdot\left(\frac{1}{2}+H\right)\left(4FF''+(F')^{2}(n-6)\right)
\end{array}\right)
\end{array}\right)
\end{array}\right), \tag{99}$$

where we can simplify as follows:

$$
\begin{aligned}
&0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\\
&=\left(\begin{array}{c}
\left(R_{\alpha\beta}-\frac{F'}{2F}\left(\begin{array}{c}
\left(\frac{\gamma^{j}_{k\beta}\gamma^{ik}_{\alpha,j}+\gamma^{j}_{k\alpha}\gamma^{ik}_{\beta,j}}{2}f_{,i}+g_{\alpha\beta}\cdot C^{ij}f_{,ij}\right)(n-2)\\
+g_{\alpha\beta}\left(\frac{\gamma^{j}_{kb}\gamma^{ik}_{a,j}+\gamma^{j}_{ka}\gamma^{ik}_{b,j}}{2}g^{ab}f_{,i}+n\cdot C^{ij}f_{,ij}\right)\\
+C^{ij}f_{,i}\left(g^{ab}g_{a\alpha}g_{\beta b,j}-n\cdot g_{\beta\alpha,j}-g^{ab}\left(g_{\alpha a}g_{\beta b,j}+g_{\beta a}g_{\alpha b,j}\right)\right)\\
+C^{ij}f_{,i}\left(\begin{array}{c}
g^{cd}\left(g_{d\beta}g_{\alpha c,j}-\frac{g_{d\beta}}{2}ng_{\alpha c,j}-\frac{g_{d\alpha}}{2}ng_{\beta c,j}\right)\\
+\frac{n^{2}}{2}g_{\alpha\beta,j}+\frac{n}{2}g_{\alpha\beta}g_{ab,j}g^{ab}
\end{array}\right)
\end{array}\right)\right)\\
-\left(R-\frac{F'}{2F}\left(\begin{array}{c}
(n-1)\left(\begin{array}{c}
2\left(\frac{\gamma^{j}_{kb}\gamma^{ik}_{a,j}+\gamma^{j}_{ka}\gamma^{ik}_{b,j}}{2}g^{ab}f_{,i}+n\cdot C^{ij}f_{,ij}\right)\\
+nC^{ij}f_{,i}g^{ab}g_{ab,j}
\end{array}\right)\\
-nC^{ij}f_{,i}g^{cd}g_{bd}g^{ab}g_{ac,j}
\end{array}\right)\right)\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\\
+\frac{g_{\alpha\beta}}{4F^{2}}C^{ij}f_{,i}f_{,j}\left(\begin{array}{c}
(F')^{2}\left(7n-6-n^{2}+n(n-1)\cdot\left(\frac{1}{2}+H\right)(n-6)\right)\\
+4FF''\left(2n(n-1)\cdot\left(\frac{1}{2}+H\right)-1\right)
\end{array}\right)
\end{array}\right),
\end{aligned}
\tag{100}
$$

$$
\begin{aligned}
&0=\left(R^{*}{}_{\alpha\beta}-R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta}\right)\\
&=\left(\begin{array}{c}
\left(R_{\alpha\beta}-\frac{F'}{2F}\left(\begin{array}{c}
\left(\frac{\gamma^{j}_{k\beta}\gamma^{ik}_{\alpha,j}+\gamma^{j}_{k\alpha}\gamma^{ik}_{\beta,j}}{2}f_{,i}+g_{\alpha\beta}\cdot C^{ij}f_{,ij}\right)(n-2)\\
+g_{\alpha\beta}\left(\frac{\gamma^{j}_{kb}\gamma^{ik}_{a,j}+\gamma^{j}_{ka}\gamma^{ik}_{b,j}}{2}g^{ab}f_{,i}+n\cdot C^{ij}f_{,ij}\right)\\
+C^{ij}f_{,i}\left(g^{ab}g_{a\alpha}g_{\beta b,j}-n\cdot g_{\beta\alpha,j}-g^{ab}\left(g_{\alpha a}g_{\beta b,j}+g_{\beta a}g_{\alpha b,j}\right)\right)\\
+C^{ij}f_{,i}\left(\begin{array}{c}
g^{cd}\left(g_{d\beta}g_{\alpha c,j}-\frac{g_{d\beta}}{2}ng_{\alpha c,j}-\frac{g_{d\alpha}}{2}ng_{\beta c,j}\right)\\
+\frac{n^{2}}{2}g_{\alpha\beta,j}+\frac{n}{2}g_{\alpha\beta}g_{ab,j}g^{ab}
\end{array}\right)
\end{array}\right)\right)\\
-\left(R-\frac{F'}{2F}\left(\begin{array}{c}
(n-1)\left(\begin{array}{c}
2\left(\frac{\gamma^{j}_{kb}\gamma^{ik}_{a,j}+\gamma^{j}_{ka}\gamma^{ik}_{b,j}}{2}g^{ab}f_{,i}+n\cdot C^{ij}f_{,ij}\right)\\
+nC^{ij}f_{,i}g^{ab}g_{ab,j}
\end{array}\right)\\
-nC^{ij}f_{,i}g^{cd}g_{bd}g^{ab}g_{ac,j}
\end{array}\right)\right)\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\\
+\frac{g_{\alpha\beta}}{8F^{2}}C^{ij}f_{,i}f_{,j}(n-1)\left(n(2H+1)-2\right)\left((F')^{2}(n-6)+4FF''\right)
\end{array}\right).
\end{aligned}
\tag{101}
$$

Fixing F[f] to the function given in (5) results in:

$$
\begin{aligned}
0 &= \left(R^{*}{}_{\alpha\beta} - R^{*}\left(\frac{1}{2}+H\right)G_{\alpha\beta} \right) \\
&= \left(\begin{array}{l}
\left(R_{\alpha\beta} - \frac{F'}{2F} \left(\begin{array}{c}
\left(\frac{\gamma^{j}_{k\beta}\gamma^{ik}_{\alpha,j} + \gamma^{j}_{k\alpha}\gamma^{ik}_{\beta,j}}{2} f_{,i} + g_{\alpha\beta}\cdot C^{ij} f_{,ij} \right)(n-2) \\
+ g_{\alpha\beta}\left(\frac{\gamma^{j}_{kb}\gamma^{ik}_{a,j} + \gamma^{j}_{ka}\gamma^{ik}_{b,j}}{2} g^{ab} f_{,i} + n\cdot C^{ij} f_{,ij} \right) \\
+ C^{ij} f_{,i}\left(g^{ab} g_{a\alpha} g_{\beta b,j} - n\cdot g_{\beta\alpha,j} - g^{ab}\left(g_{\alpha a} g_{\beta b,j} + g_{\beta a} g_{\alpha b,j} \right) \right) \\
+ C^{ij} f_{,i} \left(\begin{array}{c}
g^{cd}\left(g_{d\beta} g_{\alpha c,j} - \frac{g_{d\beta}}{2} n g_{\alpha c,j} - \frac{g_{d\alpha}}{2} n g_{\beta c,j} \right) \\
+ \frac{n^{2}}{2} g_{\alpha\beta,j} + \frac{n}{2} g_{\alpha\beta} g_{ab,j} g^{ab}
\end{array} \right)
\end{array} \right) \right) \\
- \left(R - \frac{F'}{2F} \left(\begin{array}{c}
(n-1)\left(\begin{array}{c}
2\left(\frac{\gamma^{j}_{kb}\gamma^{ik}_{a,j} + \gamma^{j}_{ka}\gamma^{ik}_{b,j}}{2} g^{ab} f_{,i} + n\cdot C^{ij} f_{,ij} \right) \\
+ nC^{ij} f_{,i} g^{ab} g_{ab,j}
\end{array} \right) \\
- nC^{ij} f_{,i} g^{cd} g_{bd} g^{ab} g_{ac,j}
\end{array} \right) \right) \cdot \left(\frac{1}{2}+H\right) g_{\alpha\beta}
\end{array} \right).
\end{aligned}
\tag{102}
$$

It was already shown elsewhere that under certain conditions and metrics this leads to the classical quantum equation of Klein-Gordon type (see for instance [5, 6], sub-section “Variation Linearity” above).

4.1 Why the Internal f-List or Spinor Structure?

As we had no need to apply the split-up option residing within (96) yet, we may wonder why bothering about such internal or spinor options at all?

Reconsidering our critical nonlinear factor $f_{,\alpha}\cdot f_{,\beta}$, where we in principle just want a substitution of the kind:

$$
f_{,\alpha}\cdot f_{,\beta} \rightarrow g_{\alpha\beta} \left\{ \begin{array}{c}
E^{Bj}_{A} f^{D}_{B\ ,j} E^{Fi}_{D} f^{A}_{F\ ,i} \\
E^{Bj}_{A} f^{D}_{B\ ,j} E^{Fi}_{D} f^{C}_{F\ ,i} \\
E^{Bj}_{A} f^{D}_{B\ ,j} E^{Fi}_{D} f^{A}_{F\ ,i}
\end{array} \right\},
\tag{103}
$$

we don’t immediately see the need for f-lists / spinors.

When trying to find such a transformation, one might start with the following approach:

$$
f_{,\alpha}\cdot f_{,\beta} = \frac{g_{ab} g^{ab}}{n} f_{,\alpha}\cdot f_{,\beta} \rightarrow g_{\alpha\beta} \frac{g^{ab}}{n} f_{,a}\cdot f_{,b}.
\tag{104}
$$

In order to be allowed to substitute the arrow by an equal sign, we either have to assume very special metrics g_{ab} and a function f, or we resort to the Dirac option with a list of functions (spinors) and demand the following:

$$f_{,\alpha}\cdot f_{,\beta} \to \begin{Bmatrix} f_{A,\alpha}\cdot f_{B,\beta} \\ f^{C}_{A,\alpha}\cdot f^{D}_{B,\beta} \\ \dots \end{Bmatrix} = g_{\alpha\beta}\begin{Bmatrix} E^{Ei}_{A} f_{E,i}\cdot E^{Fj}_{B} f_{F,j} \\ E^{Ei}_{A} f^{C}_{E,\alpha}\cdot E^{Fj}_{B} f^{D}_{F,\beta} \\ \dots \end{Bmatrix}, \tag{105}$$

where we have to find the objects E^{Ei}_{A}, E^{Fj}_{B} and the f-lists (or f-spinors) in order to fulfill the condition (105).

4.2 Dirac Split-Up Option(s)

4.2.1 The Matrix Option

Assuming the latter equation to produce eigenvalue solutions for f as follows:

$$-g_{\alpha\beta}\frac{F'}{F}M^{2}\left(f+C_{f}\right)=$$
$$=\left(\begin{array}{l}
\left(R_{\alpha\beta}-\frac{F'}{2F}\left(\begin{array}{c}
\left(\frac{\gamma^{j}_{k\beta}\gamma^{ik}_{\alpha,j}+\gamma^{j}_{k\alpha}\gamma^{ik}_{\beta,j}}{2}f_{,i}+g_{\alpha\beta}\cdot C^{ij}f_{,ij}\right)(n-2) \\
+g_{\alpha\beta}\left(\frac{\gamma^{j}_{kb}\gamma^{ik}_{a,j}+\gamma^{j}_{ka}\gamma^{ik}_{b,j}}{2}g^{ab}f_{,i}+n\cdot C^{ij}f_{,ij}\right) \\
+C^{ij}f_{,i}\left(g^{ab}g_{a\alpha}g_{\beta b,j}-n\cdot g_{\beta\alpha,j}-g^{ab}\left(g_{\alpha a}g_{\beta b,j}+g_{\beta a}g_{\alpha b,j}\right)\right) \\
+C^{ij}f_{,i}\left(\begin{array}{c}
g^{cd}\left(g_{d\beta}g_{\alpha c,j}-\frac{g_{d\beta}}{2}ng_{\alpha c,j}-\frac{g_{d\alpha}}{2}ng_{\beta c,j}\right) \\
+\frac{n^{2}}{2}g_{\alpha\beta,j}+\frac{n}{2}g_{\alpha\beta}g_{ab,j}g^{ab}
\end{array}\right)
\end{array}\right)\right) \\
-\left(R-\frac{F'}{2F}\left(\begin{array}{c}
(n-1)\left(\begin{array}{c}
2\left(\frac{\gamma^{j}_{kb}\gamma^{ik}_{a,j}+\gamma^{j}_{ka}\gamma^{ik}_{b,j}}{2}g^{ab}f_{,i}+n\cdot C^{ij}f_{,ij}\right) \\
+nC^{ij}f_{,i}g^{ab}g_{ab,j}
\end{array}\right) \\
-nC^{ij}f_{,i}g^{cd}g_{bd}g^{ab}g_{ac,j}
\end{array}\right)\right)\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}
\end{array}\right) \tag{106}$$

gives us:

$$-g_{\alpha\beta}\frac{F'}{F}M^{2}\left(f+C_{f}\right)=\frac{g_{\alpha\beta}}{8F^{2}}C^{ij}f_{,i}f_{,j}(n-1)\left(n(2H+1)-2\right)\left((F')^{2}(n-6)+4FF''\right). \tag{107}$$

Please note that the setting of $f+C_f$ does not compromise the evaluation as the constant factor would always vanish on the right-hand side of the equation above with a simple transformation of the type:

$$f\to f-C_{f}. \tag{108}$$

Now we apply the following condition again:

$$
\begin{aligned}
&\frac{4FF''+\left(F'\right)^{2}\left(n-6\right)}{4F}=F'\cdot H'\\
&\Rightarrow\quad F=\begin{cases}\left(C_{f0}+\int\limits_{1}^{f}e^{H[\phi]}\cdot\frac{n-2}{4}\cdot d\phi\right)^{\frac{4}{n-2}}\cdot C_{f1} & n>2\text{,}\\ C_{f1}\cdot e^{C_{f0}\cdot\int\limits_{1}^{f}e^{H[\phi]}d\phi} & n=2\end{cases}
\end{aligned}
\qquad (109)
$$

and solve it for the following setting H’=H’[f]=Y/f only for the cases n>2, which yields:

$$
\begin{aligned}
&\frac{4FF''+\left(F'\right)^{2}\left(n-6\right)}{4F}=F'\cdot H'=Y\cdot\frac{F'}{f}\\
&\Rightarrow\quad \begin{Bmatrix}F=C_{f1}\cdot\left(C_{f0}+f^{1+Y}\right)^{\frac{4}{n-2}} & Y\neq-1,n>2\\ F=C_{f1}\cdot\left(C_{f0}+\ln[f]\right)^{\frac{4}{n-2}} & Y=-1,n>2\end{Bmatrix}.
\end{aligned}
\qquad (110)
$$

Consequently, we obtain from (107) the following (where we already used the transformation (108)):

$$
\begin{aligned}
-g_{\alpha\beta}\frac{F'}{F}M^{2}f&=\frac{g_{\alpha\beta}}{2F}C^{ij}f_{,i}f_{,j}\left(n-1\right)\left(n\left(2H+1\right)-2\right)F'H'\\
&\Rightarrow\\
-M^{2}f&=\frac{1}{2}C^{ij}f_{,i}f_{,j}\left(n-1\right)\left(n\left(2H+1\right)-2\right)H'\quad.\\
&\Rightarrow\\
-M^{2}f^{2}&=\frac{Y}{2}C^{ij}f_{,i}f_{,j}\left(n-1\right)\left(n\left(2H+1\right)-2\right)
\end{aligned}
\qquad (111)
$$

Introducing the indices D, A for a list of functions f as shown in (95) and (96), the last line in (111) evolves to:

$$
\begin{aligned}
-M^{2}f^{2}&=\frac{Y}{2}C^{ij}f_{,i}f_{,j}\left(n-1\right)\left(n\left(2H+1\right)-2\right)\\
0&=M^{2}\mathbf{f}\cdot\mathbf{f}+Y\cdot\mathbf{E}_{A}^{j}\mathbf{E}^{Di}\mathbf{f}^{A}{}_{,j}\mathbf{f}_{D,i}\cdot\frac{\left(n-1\right)\left(n\left(2H+1\right)-2\right)}{2}\\
&=M^{2}f_{A}^{D}\cdot f_{E}^{C}+Y\cdot E_{A}^{Bj}f_{B\,,j}^{D}E_{E}^{Fi}f_{F\,,i}^{C}\cdot\frac{\left(n-1\right)\left(n\left(2H+1\right)-2\right)}{2}.\\
&\xrightarrow{m^{2}=\frac{M^{2}}{\frac{\left(n-1\right)\left(n\left(2H+1\right)-2\right)}{2}}}\\
0&=m^{2}f_{A}^{D}\cdot f_{E}^{C}+Y\cdot E_{A}^{Bj}f_{B\,,j}^{D}E_{E}^{Fi}f_{F\,,i}^{C}
\end{aligned}
\qquad (112)
$$

Thereby we have not fixed the parameter Y yet. It is clear that the product of the **E**-objects must be matrices (c.f. equations (95), (96)). Factorization leads to:

$$
\begin{aligned}
0 &= m^2 f_A^D \cdot f_E^C + Y \cdot E_A^{Bj} f^D_{B\,,j} E_E^{Fi} f^C_{F\,,i} \\
&\xrightarrow{Y=1} \\
&= m^2 f_A^D \cdot f_E^C + E_A^{Bj} f^D_{B\,,j} E_E^{Fi} f^C_{F\,,i} + i \cdot m \cdot \overbrace{\left(f_E^C E_A^{Bj} f^D_{B\,,j} - f_A^D E_E^{Fi} f^C_{F\,,i} \right)}^{=0} \\
&= \left(m \cdot f_A^D + i \cdot E_A^{Bj} f^D_{B\,,j} \right) \cdot \left(m \cdot f_E^C - i \cdot E_E^{Fi} f^C_{F\,,i} \right)
\end{aligned}
\tag{113}
$$

and gives us the Dirac-type equations:

$$
\begin{aligned}
0 &= m \cdot f_A^D + i \cdot E_A^{Bj} f^D_{B\,,j} \\
0 &= m \cdot f_E^C - i \cdot E_E^{Fi} f^C_{F\,,i}
\end{aligned}
.
\tag{114}
$$

When investigating—as an example—the situation with a metric of constants, we obtain the eigenvalue equation from (106) as follows:

$$
g_{\alpha\beta} \frac{F'}{F} M^2 f = \begin{pmatrix}
\left(\frac{F'}{2F} \begin{pmatrix} \left(\frac{\gamma^j_{k\beta} \gamma^{ik}_{\alpha,j} + \gamma^j_{k\alpha} \gamma^{ik}_{\beta,j}}{2} f_{,i} + g_{\alpha\beta} \cdot C^{ij} f_{,ij} \right)(n-2) \\ + g_{\alpha\beta} \left(\frac{\gamma^j_{kb} \gamma^{ik}_{a,j} + \gamma^j_{ka} \gamma^{ik}_{b,j}}{2} g^{ab} f_{,i} + n \cdot C^{ij} f_{,ij} \right) \end{pmatrix} \right) \\
- \frac{F'}{F}(n-1) \left(\frac{\gamma^j_{kb} \gamma^{ik}_{a,j} + \gamma^j_{ka} \gamma^{ik}_{b,j}}{2} g^{ab} f_{,i} + n \cdot C^{ij} f_{,ij} \right) \cdot \left(\frac{1}{2} + H \right) g_{\alpha\beta}
\end{pmatrix}.
\tag{115}
$$

Assuming that for a metric of constants the γ-objects in the terms $\frac{\gamma^j_{k\beta} \gamma^{ik}_{\alpha,j} + \gamma^j_{k\alpha} \gamma^{ik}_{\beta,j}}{2}$ also only contain constants, further simplification is possible and gives us:

$$
\begin{aligned}
g_{\alpha\beta} M^2 f &= \begin{pmatrix} \frac{1}{2} \left(g_{\alpha\beta} \cdot C^{ij} f_{,ij} (n-2) + g_{\alpha\beta} n \cdot C^{ij} f_{,ij} \right) \\ -(n-1) \left(n \cdot C^{ij} f_{,ij} \right) \cdot \left(\frac{1}{2} + H \right) g_{\alpha\beta} \end{pmatrix} = g_{\alpha\beta} C^{ij} f_{,ij} (n-1) \left(1 - n \cdot \left(\frac{1}{2} + H \right) \right) \\
&\Rightarrow \\
M^2 f &= -C^{ij} f_{,ij} (n-1) \left(n \left(\frac{1}{2} + H \right) - 1 \right)
\end{aligned}
.
\tag{116}
$$

Now we use the definition for m from (112), which yields:

$$
\begin{aligned}
m^2 \frac{(n-1)(n(2H+1)-2)}{2} f &= -C^{ij} f_{,ij} (n-1) \left(n \left(\frac{1}{2} + H \right) - 1 \right). \\
m^2 f = -C^{ij} f_{,ij} &\Rightarrow m^2 f + C^{ij} f_{,ij} = 0
\end{aligned}
\tag{117}
$$

Applying (95), (96), and factorizing the operator in (117) results in:

$$
\begin{gathered}
m^2 f + C^{ij} f_{,ij} = 0 \Rightarrow m^2 f_A^D + E_A^{Bj} E_F^{Di} f_{B,ij}^F = 0 \\
\Rightarrow \\
0 = m^2 f_A^D + E_A^{Bj} E_F^{Di} f_{B,ij}^F = \left(m^2 \delta_A^B \delta_F^D + E_A^{Bj} E_F^{Di} \partial_{,i} \partial_{,j} \right) f_B^F = \\
\left(m^2 \delta_A^B \delta_F^D + E_A^{Bj} E_F^{Di} \partial_{,i} \partial_{,j} + i \cdot m \cdot \overbrace{\left(\delta_F^D E_A^{Bj} \partial_{,j} - \delta_A^B E_F^{Di} \partial_{,i} \right)}^{=0} \right) f_B^F \\
= \left(m \cdot \delta_A^B + i \cdot E_A^{Bj} \partial_{,j} \right) \left(m \cdot \delta_F^D - i \cdot E_F^{Di} \partial_{,i} \right) f_B^F
\end{gathered}
\tag{118}
$$

and gives us the Dirac-type equations:

$$
\begin{aligned}
0 &= \left(m \cdot \delta_A^B + i \cdot E_A^{Bj} \partial_{,j} \right) f_B^F = m \cdot f_A^F + i \cdot E_F^{Di} \partial_i f_B^F = m \cdot f_A^F + i \cdot E_F^{Di} f_{B,i}^F \\
0 &= \left(m \cdot \delta_F^D - i \cdot E_F^{Di} \partial_{,i} \right) f_B^F = m \cdot f_B^D - i \cdot E_F^{Di} \partial_{,i} f_B^F = m \cdot f_B^D - i \cdot E_F^{Di} f_{B,i}^F
\end{aligned}
\tag{119}
$$

We recognize the classical spinor as a matrix object f_B^F. Comparison of (119) with (114):

$$
\begin{aligned}
0 = m \cdot f_A^F + i \cdot E_F^{Di} f_{B,i}^F \quad &\triangleq \quad 0 = m \cdot f_A^D + i \cdot E_A^{Bj} f_{B,j}^D \\
0 = m \cdot f_B^D - i \cdot E_F^{Di} f_{B,i}^F \quad &\triangleq \quad 0 = m \cdot f_E^C - i \cdot E_E^{Fi} f_{F,i}^C
\end{aligned}
\tag{120}
$$

shows perfect structural agreement and just leaves us with the quest of finding the right mathematical objects E_F^{Di}, where we already identified the Dirac matrices as one possible option.

4.2.2 The Classical Spinor or f-List Option

For nostalgic reasons one might be interested in insisting on simple lists of f and so we rewrite (95), (96) as follows:

$$
f_{A,\alpha} \cdot f_{B,\beta} = g_{\alpha\beta} E_A^{Ei} f_{E,i} \cdot E_B^{Fj} f_{F,j} . \tag{121}
$$

"In effectivo"—so one might say—we have substituted the list of derivatives of a scalar function f by a list of derivatives of a list of functions f_A in such a way that the metric tensor can be factored out.
The subsequent evaluation (112) then becomes:

$$
\begin{gathered}
-M^2 f^2 = \frac{Y}{2} C^{ij} f_{,i} f_{,j} (n-1)\left(n(2H+1)-2\right) \\
0 = M^2 f_A \cdot f_B + Y \cdot E_A^{Ei} f_{E,i} \cdot E_B^{Fj} f_{F,j} \cdot \frac{(n-1)\left(n(2H+1)-2\right)}{2} . \\
\xrightarrow{m^2 = \frac{M^2}{\frac{(n-1)(n(2H+1)-2)}{2}}} \\
0 = m^2 f_A \cdot f_B + Y \cdot E_A^{Ei} f_{E,i} \cdot E_B^{Fj} f_{F,j}
\end{gathered}
\tag{122}
$$

The **E**-objects must still be matrices. The factorization leads to:

$$
\begin{gathered}
0 = m^2 f_A \cdot f_B + Y \cdot E_A^{Ei} f_{E,i} \cdot E_B^{Fj} f_{F,j} \\
\xrightarrow{Y=1} \\
= m^2 f_A \cdot f_B + Y \cdot E_A^{Ei} f_{E,i} \cdot E_B^{Fj} f_{F,j} + i \cdot m \cdot \overbrace{\left(f_B \cdot E_A^{Ej} f_{E,j} - f_A \cdot E_B^{Fj} f_{F,j} \right)}^{=0} \\
= \left(m \cdot f_A + i \cdot E_A^{Ej} f_{E,j} \right) \cdot \left(m \cdot f_B - i \cdot E_B^{Fi} f_{F,i} \right)
\end{gathered}
\tag{123}
$$

and gives us the new—“mass-spinor-like”—Dirac-type equations:

$$\begin{aligned} 0 &= m\cdot f_A + i\cdot E_A^{Ej} f_{E,j} \\ 0 &= m\cdot f_B - i\cdot E_B^{Fi} f_{F,i} \end{aligned}. \tag{124}$$

It should be noted that the factorization in (123) requires us to demand the permutability of the indices A and B in the term $\left(f_B\cdot E_A^{Ej} f_{E,j} - f_A\cdot E_B^{Fj} f_{F,j}\right)$.

The corresponding evaluation from (118) then changes to:

$$\begin{aligned} & m^2 f_A + C^{ij} f_{A,ij} = 0 \Rightarrow m^2\cdot f_A + E_A^{Bj} E_B^{Ci} f_{C,ij} = 0 \\ & \Rightarrow \\ & 0 = m^2\cdot f_A + E_A^{Bj} E_B^{Ci} f_{C,ij} = \left(m^2 \delta_A^B \delta_B^C + E_A^{Bj} E_B^{Ci} \partial_{,i} \partial_{,j}\right) f_C \\ & = \left(m^2 \delta_A^B \delta_B^C + E_A^{Bj} E_B^{Ci} \partial_{,i} \partial_{,j} + i\cdot m\cdot \overbrace{\left(\delta_B^C E_A^{Bj} \partial_{,j} - \delta_A^B E_B^{Ci} \partial_{,i}\right)}^{=0}\right) f_C \\ & = \left(m\cdot \delta_A^B + i\cdot E_A^{Bj} \partial_{,j}\right)\left(m\cdot \delta_B^C - i\cdot E_B^{Ci} \partial_{,i}\right) f_C \end{aligned}, \tag{125}$$

and—as we can interchange the signs of the two factors / operators:

$$0 = \left(m\cdot \delta_A^B + i\cdot E_A^{Bj} \partial_{,j}\right)\left(m\cdot \delta_B^C - i\cdot E_B^{Ci} \partial_{,i}\right) f_C = \left(m\cdot \delta_A^B - i\cdot E_A^{Bj} \partial_{,j}\right)\left(m\cdot \delta_B^C + i\cdot E_B^{Ci} \partial_{,i}\right) f_C, \tag{126}$$

this gives us the Dirac-type equations:

$$\begin{aligned} 0 &= \left(m\cdot \delta_B^C - i\cdot E_B^{Ci} \partial_{,i}\right) f_C \\ 0 &= \left(m\cdot \delta_B^C + i\cdot E_B^{Ci} \partial_{,i}\right) f_C \end{aligned}. \tag{127}$$

Comparison of (124) with (125) shows perfect structural agreement:

$$\begin{aligned} 0 &= \left(m\cdot \delta_B^C - i\cdot E_B^{Ci} \partial_{,i}\right) f_C = m\cdot f_B - i\cdot E_B^{Cj} f_{C,j} = m\cdot f_B - i\cdot E_B^{Fi} f_{F,i} \\ 0 &= \left(m\cdot \delta_B^C + i\cdot E_B^{Ci} \partial_{,i}\right) f_C = m\cdot f_B + i\cdot E_B^{Ej} f_{E,j} = m\cdot f_A + i\cdot E_A^{Ej} f_{E,j} \end{aligned} \tag{128}$$

and, just as before with the matrix-like f-spinors, leaves us with the quest of finding the right mathematical objects E_F^{Di}, where we already identified the Dirac matrices as one possible option.

4.2.3 The Introduction of “Mass Spinors”

Insisting on f remaining just a scalar function, we rewrite (95), (96) as follows:

$$f_{,\alpha}\cdot f_{,\beta} = g_{\alpha\beta} E_A^{Bj} f_{,j} E_B^{Ai} f_{,i}. \tag{129}$$

The subsequent evaluation (112) then becomes:

$$\begin{aligned} & -M^2 f^2 = \frac{Y}{2} C^{ij} f_{,i} f_{,j} (n-1)(n(2H+1)-2) \\ & 0 = M^2 f\cdot f + Y\cdot E_A^{Bj} f_{,j} E_B^{Ai} f_{,i}\cdot \frac{(n-1)(n(2H+1)-2)}{2} \\ & \xrightarrow{m^2 = \frac{M^2}{\frac{(n-1)(n(2H+1)-2)}{2}}} \\ & 0 = m^2 f\cdot f + Y\cdot E_A^{Bj} f_{,j} E_B^{Ai} f_{,i} \end{aligned}. \tag{130}$$

As already said before (c.f. equations (95), (96)), it is clear that the **E**-objects must be matrices. It is also clear that this time, with f being a scalar, the matrix-character has to be taken on by the masses. Thus, factorization leads to:

$$\begin{aligned}
0 &= m^2 f\cdot f + Y\cdot E_A^{Bj} f_{,j} E_B^{Ai} f_{,i} = m_A^B\cdot m_B^A\cdot f\cdot f + Y\cdot E_A^{Bj} f_{,j} E_B^{Ai} f_{,i} \\
&\xrightarrow{Y=1} \\
&= m_A^B\cdot m_B^A\cdot f\cdot f + E_A^{Bj} f_{,j} E_B^{Ai} f_{,i} + i\cdot\overbrace{\left(m_B^A\cdot f\cdot E_A^{Bj} f_{,j} - m_A^B\cdot f\cdot E_B^{Ai} f_{,i}\right)}^{=0} \\
&= \left(m_A^B\cdot f + i\cdot E_A^{Bj} f_{,j}\right)\cdot\left(m_B^A\cdot f - i\cdot E_B^{Ai} f_{,i}\right)
\end{aligned} \tag{131}$$

and gives us the new—"mass-spinor-like"—Dirac-type equations:

$$\begin{aligned}
0 &= m_A^B\cdot f + i\cdot E_A^{Bj} f_{,j} \\
0 &= m_B^A\cdot f - i\cdot E_B^{Ai} f_{,i}
\end{aligned}. \tag{132}$$

The corresponding evaluation from (118) then changes to:

$$\begin{aligned}
& m^2 f + C^{ij} f_{,ij} = 0 \Rightarrow m_A^B\cdot m_B^A\cdot f + E_A^{Bj} E_B^{Ai} f_{,ij} = 0 \\
& \Rightarrow \\
0 &= m_A^B\cdot m_B^A\cdot f + E_A^{Bj} E_B^{Ai} f_{,ij} = \left(m_A^B\cdot m_B^A + E_A^{Bj} E_B^{Ai}\partial_{,i}\partial_{,j}\right) f \\
&= \left(m_A^B\cdot m_B^A + E_A^{Bj} E_B^{Ai}\partial_{,i}\partial_{,j} + i\cdot\overbrace{\left(m_B^A E_A^{Bj}\partial_{,j} - m_A^B E_B^{Ai}\partial_{,i}\right)}^{=0}\right) f \\
&= \left(m_A^B + i\cdot E_A^{Bj}\partial_{,j}\right)\left(m_B^A - i\cdot E_B^{Ai}\partial_{,i}\right) f
\end{aligned} \tag{133}$$

and—as we can interchange the two factors / operators:

$$0 = \left(m_A^B + i\cdot E_A^{Bj}\partial_{,j}\right)\left(m_B^A\cdot f - i\cdot E_B^{Ai}\partial_{,i}\right) f = \left(m_B^A\cdot f - i\cdot E_B^{Ai}\partial_{,i}\right)\left(m_A^B + i\cdot E_A^{Bj}\partial_{,j}\right) f, \tag{134}$$

gives us the Dirac-type equations:

$$\begin{aligned}
0 &= \left(m_A^B + i\cdot E_A^{Bj}\partial_{,j}\right) f \\
0 &= \left(m_B^A - i\cdot E_B^{Ai}\partial_{,i}\right) f
\end{aligned}. \tag{135}$$

This time the mass takes on the character of the spinor in the form of a matrix object. Comparison of (132) with (135) shows perfect structural agreement and, just as before with the f-spinors, leaves us with the quest of finding the right mathematical objects E_F^{Di}, where we already identified the Dirac matrices as one possible option.

4.3 The Other Motivation for the Dirac Theory

The usual justification for the introduction of the Dirac theory via a factorization of the Klein-Gordon equation resulted from the probability interpretation of Quantum Theory and certain negative probability densities. Here we saw that the Dirac technique is needed for the linearization of the quantum gravity field equations, leading to additive results with respect to the metric volume factors. In other words, we now have a metric justification for something which originally was postulated in order to get rid of some technical difficulties and impossible interpretations.

Demanding additivity and thus, infinite dimensionality, we require linearity. The latter forces us to develop a generalized Dirac concept.

4.4 Generalization— “Transformers”

Apart from the fact that we could easily generalize the considerations above to an arbitrary (non-constant) metric tensor $g_{\alpha\beta}$, we here intend to go even further and introduce metric transformations beyond the volume scaling, which means something of the type:

$$G_{\alpha\beta} = \tau_{\alpha}^{\nu} \cdot g_{\nu\beta} \tag{136}$$

or even:

$$G_{\alpha\beta} = \tau_{\alpha\beta}^{\mu\nu} \cdot g_{\mu\nu} \,. \tag{137}$$

Thereby the transformation objects or “transformers” can be arbitrary objects, but they should still assure $G_{\alpha\beta}$ to be a metric tensor with the usual tensor properties regarding symmetry and coordinate transformations.

When now, just to give an example, demanding that $G_{\alpha\beta}$ shall be a metric of constants (perhaps even with a volumetric scaling factor “Const”):

$$G_{\alpha\beta} = \mathrm{Const} \cdot \begin{pmatrix} \mathrm{const}_{00} & \mathrm{const}_{01} & \cdots \\ \mathrm{const}_{01} & \mathrm{const}_{11} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{138}$$

we would find the quantum gravity field equations (2) already being fulfilled even in their generalized form. Things of interest could still be happening in the level underneath, which is to say on the level of the metric $g_{\alpha\beta}$ and the transformers τ_{α}^{ν} and $\tau_{\alpha\beta}^{\mu\nu}$.

For the derivation of the corresponding field equations we have to evaluate the corresponding contravariant metric tensors of (136) and (137) first. Those would have to be:

$$G_{\alpha\beta} G^{\beta\gamma} = \delta_{\alpha}^{\gamma} \quad \Rightarrow \quad \tau_{\alpha}^{\nu} \cdot g_{\nu\beta} G^{\beta\gamma} = \delta_{\alpha}^{\gamma} \quad \Rightarrow \quad G^{\beta\gamma} = \tau^{-1\gamma}_{\ \ \ \mu} \cdot g^{\beta\mu} \tag{139}$$

and:

$$G_{\alpha\beta} G^{\beta\gamma} = \delta_{\alpha}^{\gamma} \quad \Rightarrow \quad \tau_{\alpha\beta}^{\mu\nu} \cdot g_{\mu\nu} G^{\beta\gamma} = \delta_{\alpha}^{\gamma} \quad \Rightarrow \quad G^{\beta\gamma} = \tau^{-1\beta\gamma}_{\ \ \ \ \chi\delta} \cdot g^{\chi\delta} \,, \tag{140}$$

where the terms $\tau^{-1\nu}_{\ \ \ \alpha}$, $\tau^{-1\mu\nu}_{\ \ \ \ \alpha\beta}$ just define the inverse functions of τ_{α}^{ν} and $\tau_{\alpha\beta}^{\mu\nu}$, respectively.

The corresponding Ricci tensors can be evaluated as follows:

$$
\begin{aligned}
R^{*}{}_{\alpha\beta} &= \left(\begin{array}{l}
-\frac{1}{2}\left(G_{\alpha\beta,ab}+G_{ab,\alpha\beta}-G_{\alpha b,a\beta}-G_{\beta b,a\alpha}\right)G^{ab} \\
+\frac{1}{2}\left(\frac{1}{2}G_{ac,\alpha}\cdot G_{bd,\beta}+G_{\alpha c,a}\cdot G_{\beta d,b}-G_{\alpha c,a}\cdot G_{\beta b,d}\right)G^{ab}G^{cd} \\
-\frac{1}{4}\left(G_{\beta c,\alpha}+G_{\alpha c,\beta}-G_{\alpha\beta,c}\right)\left(2G_{bd,a}-G_{ab,d}\right)G^{ab}G^{cd}
\end{array}\right) \\
&= \left(\begin{array}{c}
-\frac{\tau^{-1a}{}_{\chi}}{2}\left(\begin{array}{l}
\tau^{\mu}_{\alpha}\cdot g_{\mu\beta,ab}+\tau^{\mu}_{\alpha,b}g_{\mu\beta,a}+\tau^{\mu}_{\alpha,a}g_{\mu\beta,b}+\tau^{\mu}_{\alpha,ab}g_{\mu\beta} \\
+\tau^{\mu}_{a}\cdot g_{\mu b,\alpha\beta}+\tau^{\mu}_{a,\alpha\beta}g_{\mu b}+\tau^{\mu}_{a,\alpha}g_{\mu b,\beta}+\tau^{\mu}_{a,\beta}g_{\mu b,\alpha} \\
-\tau^{\mu}_{\alpha}\cdot g_{\mu b,a\beta}-\tau^{\mu}_{\alpha,a\beta}g_{\mu b}-\tau^{\mu}_{\alpha,a}g_{\mu b,\beta}-\tau^{\mu}_{\alpha,\beta}g_{\mu b,a} \\
-\tau^{\mu}_{\beta}\cdot g_{\mu b,a\alpha}-\tau^{\mu}_{\beta,a\alpha}g_{\mu b}-\tau^{\mu}_{\beta,a}g_{\mu b,\alpha}-\tau^{\mu}_{\beta,\alpha}g_{\mu b,a}
\end{array}\right)g^{\chi b} \\
+\frac{\tau^{-1a}{}_{\chi}\tau^{-1c}{}_{\gamma}}{2}\left(\begin{array}{l}
\frac{1}{2}\left(\tau^{\mu}_{a}\cdot g_{\mu c,\alpha}+\tau^{\mu}_{a,\alpha}g_{\mu c}\right)\cdot\left(\tau^{\mu}_{b}\cdot g_{\mu d,\beta}+\tau^{\mu}_{b,\beta}g_{\mu d}\right) \\
+\left(\tau^{\mu}_{\alpha}\cdot g_{\mu c,a}+\tau^{\mu}_{\alpha,a}g_{\mu c}\right)\cdot\left(\tau^{\mu}_{\beta}\cdot g_{\mu d,b}+\tau^{\mu}_{\beta,b}g_{\mu d}\right) \\
-\left(\tau^{\mu}_{\alpha}\cdot g_{\mu c,a}+\tau^{\mu}_{\alpha,a}g_{\mu c}\right)\cdot\left(\tau^{\mu}_{\beta}\cdot g_{\mu b,d}+\tau^{\mu}_{\beta,d}g_{\mu b}\right)
\end{array}\right)g^{\chi b}g^{\gamma d} \\
-\frac{\tau^{-1a}{}_{\chi}\tau^{-1c}{}_{\gamma}}{4}\left(\begin{array}{l}
\tau^{\mu}_{\alpha}\cdot g_{\mu c,\beta}+\tau^{\mu}_{\alpha,\beta}g_{\mu c}+\tau^{\mu}_{\beta}\cdot g_{\mu c,\alpha} \\
+\tau^{\mu}_{\beta,\alpha}g_{\mu c}-\tau^{\mu}_{\alpha}\cdot g_{\mu\beta,c}-\tau^{\mu}_{\alpha,c}g_{\mu\beta}
\end{array}\right) \\
\times\left(2\left(\tau^{\mu}_{b}\cdot g_{\mu d,a}+\tau^{\mu}_{b,a}g_{\mu d}\right)-\tau^{\mu}_{a}\cdot g_{\mu b,d}-\tau^{\mu}_{a,d}g_{\mu b}\right)g^{\chi b}g^{\gamma d}
\end{array}\right),
\end{aligned}
\tag{141}
$$

$$
R^{*}{}_{\alpha\beta} = \left(\begin{array}{c}
-\frac{\tau^{-1ab}{}_{\chi\delta}}{2}\left(\begin{array}{l}
\tau^{\mu\nu}_{\alpha\beta}\cdot g_{\mu\nu,ab}+\tau^{\mu\nu}_{\alpha\beta,b}g_{\mu\nu,a}+\tau^{\mu\nu}_{\alpha\beta,a}g_{\mu\nu,b}+\tau^{\mu\nu}_{\alpha\beta,ab}g_{\mu\nu} \\
+\tau^{\mu\nu}_{ab}\cdot g_{\mu\nu,\alpha\beta}+\tau^{\mu\nu}_{ab,\alpha\beta}g_{\mu\nu}+\tau^{\mu\nu}_{ab,\alpha}g_{\mu\nu,\beta}+\tau^{\mu\nu}_{ab,\beta}g_{\mu\nu,\alpha} \\
-\tau^{\mu\nu}_{\alpha b}\cdot g_{\mu\nu,a\beta}-\tau^{\mu\nu}_{\alpha b,a\beta}g_{\mu\nu}-\tau^{\mu\nu}_{\alpha b,a}g_{\mu\nu,\beta}-\tau^{\mu\nu}_{\alpha b,\beta}g_{\mu\nu,a} \\
-\tau^{\mu\nu}_{\beta b}\cdot g_{\mu\nu,a\alpha}-\tau^{\mu\nu}_{\beta b,a\alpha}g_{\mu\nu}-\tau^{\mu\nu}_{\beta b,a}g_{\mu\nu,\alpha}-\tau^{\mu\nu}_{\beta b,\alpha}g_{\mu\nu,a}
\end{array}\right)g^{\chi\delta} \\
+\frac{\tau^{-1ab}{}_{\chi\delta}\tau^{-1cd}{}_{\gamma\eta}}{2}\left(\begin{array}{l}
\frac{1}{2}\left(\tau^{\mu\nu}_{ac}\cdot g_{\mu\nu,\alpha}+\tau^{\mu\nu}_{ac,\alpha}g_{\mu\nu}\right)\cdot\left(\tau^{\mu\nu}_{bd}\cdot g_{\mu\nu,\beta}+\tau^{\mu\nu}_{bd,\beta}g_{\mu\nu}\right) \\
+\left(\tau^{\mu\nu}_{\alpha c}\cdot g_{\mu\nu,a}+\tau^{\mu\nu}_{\alpha c,a}g_{\mu\nu}\right)\cdot\left(\tau^{\mu\nu}_{\beta d}\cdot g_{\mu\nu,b}+\tau^{\mu\nu}_{\beta d,b}g_{\mu\nu}\right) \\
-\left(\tau^{\mu\nu}_{\alpha c}\cdot g_{\mu\nu,a}+\tau^{\mu\nu}_{\alpha c,a}g_{\mu\nu}\right)\cdot\left(\tau^{\mu\nu}_{\beta b}\cdot g_{\mu\nu,d}+\tau^{\mu\nu}_{\beta b,d}g_{\mu\nu}\right)
\end{array}\right)g^{\chi\delta}g^{\gamma\eta} \\
-\frac{\tau^{-1ab}{}_{\chi\delta}\tau^{-1cd}{}_{\gamma\eta}}{4}\left(\begin{array}{l}
\tau^{\mu\nu}_{\alpha c}\cdot g_{\mu\nu,\beta}+\tau^{\mu\nu}_{\alpha c,\beta}g_{\mu\nu}+\tau^{\mu\nu}_{\beta c}\cdot g_{\mu\nu,\alpha} \\
+\tau^{\mu\nu}_{\beta c,\alpha}g_{\mu\nu}-\tau^{\mu\nu}_{\alpha\beta}\cdot g_{\mu\nu,c}-\tau^{\mu\nu}_{\alpha\beta,c}g_{\mu\nu}
\end{array}\right) \\
\times\left(2\left(\tau^{\mu\nu}_{bd}\cdot g_{\mu\nu,a}+\tau^{\mu\nu}_{bd,a}g_{\mu\nu}\right)-\tau^{\mu\nu}_{ab}\cdot g_{\mu\nu,d}-\tau^{\mu\nu}_{ab,d}g_{\mu\nu}\right)g^{\chi\delta}g^{\gamma\eta}
\end{array}\right).
\tag{142}
$$

The Ricci scalar, which we find via:

$$
R^{*}{}_{\alpha\beta}G^{\alpha\beta}=\left(\begin{array}{l}
-\frac{1}{2}\left(G_{\alpha\beta,ab}+G_{ab,\alpha\beta}-G_{\alpha b,a\beta}-G_{\beta b,a\alpha}\right)G^{ab}\\
+\frac{1}{2}\left(\frac{1}{2}G_{ac,\alpha}\cdot G_{bd,\beta}+G_{\alpha c,a}\cdot G_{\beta d,b}-G_{\alpha c,a}\cdot G_{\beta b,d}\right)G^{ab}G^{cd}\\
-\frac{1}{4}\left(G_{\beta c,\alpha}+G_{\alpha c,\beta}-G_{\alpha\beta,c}\right)\left(2G_{bd,a}-G_{ab,d}\right)G^{ab}G^{cd}
\end{array}\right)
$$

$$
=\left(\begin{array}{c}
-\frac{\tau^{-1a}{}_{\chi}}{2}\left(\begin{array}{l}
\tau^{\mu}_{\alpha}\cdot g_{\mu\beta,ab}+\tau^{\mu}_{\alpha,b}g_{\mu\beta,a}+\tau^{\mu}_{\alpha,a}g_{\mu\beta,b}+\tau^{\mu}_{\alpha,ab}g_{\mu\beta}\\
+\tau^{\mu}_{a}\cdot g_{\mu b,\alpha\beta}+\tau^{\mu}_{a,\alpha\beta}g_{\mu b}+\tau^{\mu}_{a,\alpha}g_{\mu b,\beta}+\tau^{\mu}_{a,\beta}g_{\mu b,\alpha}\\
-\tau^{\mu}_{\alpha}\cdot g_{\mu b,a\beta}-\tau^{\mu}_{\alpha,a\beta}g_{\mu b}-\tau^{\mu}_{\alpha,a}g_{\mu b,\beta}-\tau^{\mu}_{\alpha,\beta}g_{\mu b,a}\\
-\tau^{\mu}_{\beta}\cdot g_{\mu b,a\alpha}-\tau^{\mu}_{\beta,a\alpha}g_{\mu b}-\tau^{\mu}_{\beta,a}g_{\mu b,\alpha}-\tau^{\mu}_{\beta,\alpha}g_{\mu b,a}
\end{array}\right)g^{\chi b}\\
+\frac{\tau^{-1a}{}_{\chi}\tau^{-1c}{}_{\gamma}}{2}\left(\begin{array}{l}
\frac{1}{2}\left(\tau^{\mu}_{a}\cdot g_{\mu c,\alpha}+\tau^{\mu}_{a,\alpha}g_{\mu c}\right)\cdot\left(\tau^{\mu}_{b}\cdot g_{\mu d,\beta}+\tau^{\mu}_{b,\beta}g_{\mu d}\right)\\
+\left(\tau^{\mu}_{\alpha}\cdot g_{\mu c,a}+\tau^{\mu}_{\alpha,a}g_{\mu c}\right)\cdot\left(\tau^{\mu}_{\beta}\cdot g_{\mu d,b}+\tau^{\mu}_{\beta,b}g_{\mu d}\right)\\
-\left(\tau^{\mu}_{\alpha}\cdot g_{\mu c,a}+\tau^{\mu}_{\alpha,a}g_{\mu c}\right)\cdot\left(\tau^{\mu}_{\beta}\cdot g_{\mu b,d}+\tau^{\mu}_{\beta,d}g_{\mu b}\right)
\end{array}\right)g^{\chi b}g^{\gamma d}\\
-\frac{\tau^{-1a}{}_{\chi}\tau^{-1c}{}_{\gamma}}{4}\left(\begin{array}{l}
\tau^{\mu}_{\alpha}\cdot g_{\mu c,\beta}+\tau^{\mu}_{\alpha,\beta}g_{\mu c}+\tau^{\mu}_{\beta}\cdot g_{\mu c,\alpha}\\
+\tau^{\mu}_{\beta,\alpha}g_{\mu c}-\tau^{\mu}_{\alpha}\cdot g_{\mu\beta,c}-\tau^{\mu}_{\alpha,c}g_{\mu\beta}
\end{array}\right)\\
\times\left(2\left(\tau^{\mu}_{b}\cdot g_{\mu d,a}+\tau^{\mu}_{b,a}g_{\mu d}\right)-\tau^{\mu}_{a}\cdot g_{\mu b,d}-\tau^{\mu}_{a,d}g_{\mu b}\right)g^{\chi b}g^{\gamma d}
\end{array}\right)\tau^{-1\beta}{}_{\kappa}g^{\alpha\kappa}\quad, \qquad (143)
$$

$$
R^{*}=R^{*}{}_{\alpha\beta}G^{\alpha\beta}
$$

$$
=\left(\begin{array}{c}
-\frac{\tau^{-1ab}{}_{\chi\delta}}{2}\left(\begin{array}{l}
\tau^{\mu\nu}_{\alpha\beta}\cdot g_{\mu\nu,ab}+\tau^{\mu\nu}_{\alpha\beta,b}g_{\mu\nu,a}+\tau^{\mu\nu}_{\alpha\beta,a}g_{\mu\nu,b}+\tau^{\mu\nu}_{\alpha\beta,ab}g_{\mu\nu}\\
+\tau^{\mu\nu}_{ab}\cdot g_{\mu\nu,\alpha\beta}+\tau^{\mu\nu}_{ab,\alpha\beta}g_{\mu\nu}+\tau^{\mu\nu}_{ab,\alpha}g_{\mu\nu,\beta}+\tau^{\mu\nu}_{ab,\beta}g_{\mu\nu,\alpha}\\
-\tau^{\mu\nu}_{\alpha b}\cdot g_{\mu\nu,a\beta}-\tau^{\mu\nu}_{\alpha b,a\beta}g_{\mu\nu}-\tau^{\mu\nu}_{\alpha b,a}g_{\mu\nu,\beta}-\tau^{\mu\nu}_{\alpha b,\beta}g_{\mu\nu,a}\\
-\tau^{\mu\nu}_{\beta b}\cdot g_{\mu\nu,a\alpha}-\tau^{\mu\nu}_{\beta b,a\alpha}g_{\mu\nu}-\tau^{\mu\nu}_{\beta b,a}g_{\mu\nu,\alpha}-\tau^{\mu\nu}_{\beta b,\alpha}g_{\mu\nu,a}
\end{array}\right)g^{\chi\delta}\\
+\frac{\tau^{-1ab}{}_{\chi\delta}\tau^{-1cd}{}_{\gamma\eta}}{2}\left(\begin{array}{l}
\frac{1}{2}\left(\tau^{\mu\nu}_{ac}\cdot g_{\mu\nu,\alpha}+\tau^{\mu\nu}_{ac,\alpha}g_{\mu\nu}\right)\cdot\left(\tau^{\mu\nu}_{bd}\cdot g_{\mu\nu,\beta}+\tau^{\mu\nu}_{bd,\beta}g_{\mu\nu}\right)\\
+\left(\tau^{\mu\nu}_{\alpha c}\cdot g_{\mu\nu,a}+\tau^{\mu\nu}_{\alpha c,a}g_{\mu\nu}\right)\cdot\left(\tau^{\mu\nu}_{\beta d}\cdot g_{\mu\nu,b}+\tau^{\mu\nu}_{\beta d,b}g_{\mu\nu}\right)\\
-\left(\tau^{\mu\nu}_{\alpha c}\cdot g_{\mu\nu,a}+\tau^{\mu\nu}_{\alpha c,a}g_{\mu\nu}\right)\cdot\left(\tau^{\mu\nu}_{\beta b}\cdot g_{\mu\nu,d}+\tau^{\mu\nu}_{\beta b,d}g_{\mu\nu}\right)
\end{array}\right)g^{\chi\delta}g^{\gamma\eta}\\
-\frac{\tau^{-1ab}{}_{\chi\delta}\tau^{-1cd}{}_{\gamma\eta}}{4}\left(\begin{array}{l}
\tau^{\mu\nu}_{\alpha c}\cdot g_{\mu\nu,\beta}+\tau^{\mu\nu}_{\alpha c,\beta}g_{\mu\nu}+\tau^{\mu\nu}_{\beta c}\cdot g_{\mu\nu,\alpha}\\
+\tau^{\mu\nu}_{\beta c,\alpha}g_{\mu\nu}-\tau^{\mu\nu}_{\alpha\beta}\cdot g_{\mu\nu,c}-\tau^{\mu\nu}_{\alpha\beta,c}g_{\mu\nu}
\end{array}\right)\\
\times\left(2\left(\tau^{\mu\nu}_{bd}\cdot g_{\mu\nu,a}+\tau^{\mu\nu}_{bd,a}g_{\mu\nu}\right)-\tau^{\mu\nu}_{ab}\cdot g_{\mu\nu,d}-\tau^{\mu\nu}_{ab,d}g_{\mu\nu}\right)g^{\chi\delta}g^{\gamma\eta}
\end{array}\right)\tau^{-1\alpha\beta}{}_{\lambda\kappa}g^{\lambda\kappa}\quad, \qquad (144)
$$

requires some considerations with respect to expressions like:

$$
\tau^{-1ab}{}_{\chi\delta}g^{\chi\delta}\tau^{\mu\nu}_{\alpha\beta}\cdot g_{\mu\nu,ab}\tau^{-1\alpha\beta}{}_{\lambda\kappa}g^{\lambda\kappa}=\tau^{-1ab}{}_{\chi\delta}g^{\chi\delta}\cdot g_{\mu\nu,ab}\delta^{\mu}_{\lambda}\delta^{\nu}_{\kappa}g^{\lambda\kappa}=\tau^{-1ab}{}_{\chi\delta}g^{\chi\delta}\cdot g_{\mu\nu,ab}g^{\mu\nu}, \qquad (145)
$$

in order to exploit the full potential of simplification. As it does not provide much, however, we leave it to the interested reader to perform all such derivations. Here we are now—as before in this paper and the book [5]—interested in the cases of metrics of constants, where we obtain:

$$R^{*}{}_{\alpha\beta}=\begin{pmatrix} -\frac{\tau^{-1a}{}_{\chi}}{2}\left(\tau^{\mu}_{\alpha,ab}g_{\mu\beta}+\tau^{\mu}_{a,\alpha\beta}g_{\mu b}-\tau^{\mu}_{\alpha,a\beta}g_{\mu b}-\tau^{\mu}_{\beta,a\alpha}g_{\mu b}\right)g^{\chi b} \\ +\frac{\tau^{-1a}{}_{\chi}\tau^{-1c}{}_{\gamma}}{2}\left(\frac{1}{2}\tau^{\mu}_{a,\alpha}g_{\mu c}\cdot\tau^{\xi}_{b,\beta}g_{\xi d}+\tau^{\mu}_{\alpha,a}g_{\mu c}\cdot\tau^{\xi}_{\beta,b}g_{\xi d}-\tau^{\mu}_{\alpha,a}g_{\mu c}\cdot\tau^{\xi}_{\beta,d}g_{\xi b}\right)g^{\chi b}g^{\gamma d} \\ -\frac{\tau^{-1a}{}_{\chi}\tau^{-1c}{}_{\gamma}}{4}\left(\tau^{\mu}_{\alpha,\beta}g_{\mu c}+\tau^{\mu}_{\beta,\alpha}g_{\mu c}-\tau^{\mu}_{\alpha,c}g_{\mu\beta}\right)\left(2\tau^{\xi}_{b,a}g_{\xi d}-\tau^{\xi}_{a,d}g_{\xi b}\right)g^{\chi b}g^{\gamma d} \end{pmatrix}, \quad (146)$$

$$R^{*}{}_{\alpha\beta}=\begin{pmatrix} -\frac{\tau^{-1ab}{}_{\chi\delta}}{2}\left(\tau^{\mu\nu}_{\alpha\beta,ab}g_{\mu\nu}+\tau^{\mu\nu}_{ab,\alpha\beta}g_{\mu\nu}-\tau^{\mu\nu}_{\alpha b,a\beta}g_{\mu\nu}-\tau^{\mu\nu}_{\beta b,a\alpha}g_{\mu\nu}\right)g^{\chi\delta} \\ +\frac{\tau^{-1ab}{}_{\chi\delta}\tau^{-1cd}{}_{\gamma\eta}}{2}\left(\frac{1}{2}\tau^{\mu\nu}_{ac,\alpha}g_{\mu\nu}\cdot\tau^{\xi\zeta}_{bd,\beta}g_{\xi\zeta}+\tau^{\mu\nu}_{\alpha c,a}g_{\mu\nu}\cdot\tau^{\xi\zeta}_{\beta d,b}g_{\xi\zeta}-\tau^{\mu\nu}_{\alpha c,a}g_{\mu\nu}\cdot\tau^{\xi\zeta}_{\beta b,d}g_{\xi\zeta}\right)g^{\chi\delta}g^{\gamma\eta} \\ -\frac{\tau^{-1ab}{}_{\chi\delta}\tau^{-1cd}{}_{\gamma\eta}}{4}\left(\tau^{\mu\nu}_{\alpha c,\beta}g_{\mu\nu}+\tau^{\mu\nu}_{\beta c,\alpha}g_{\mu\nu}-\tau^{\mu\nu}_{\alpha\beta,c}g_{\mu\nu}\right)\left(2\tau^{\xi\zeta}_{bd,a}g_{\xi\zeta}-\tau^{\xi\zeta}_{ab,d}g_{\xi\zeta}\right)g^{\chi\delta}g^{\gamma\eta} \end{pmatrix}. \quad (147)$$

$$=g_{\mu\nu}g^{\chi\delta}\frac{\tau^{-1ab}{}_{\chi\delta}}{2}\begin{pmatrix} -\left(\tau^{\mu\nu}_{\alpha\beta,ab}+\tau^{\mu\nu}_{ab,\alpha\beta}-\tau^{\mu\nu}_{\alpha b,a\beta}-\tau^{\mu\nu}_{\beta b,a\alpha}\right) \\ +\tau^{-1cd}{}_{\gamma\eta}\left(\frac{1}{2}\tau^{\mu\nu}_{ac,\alpha}\cdot\tau^{\xi\zeta}_{bd,\beta}+\tau^{\mu\nu}_{\alpha c,a}\cdot\tau^{\xi\zeta}_{\beta d,b}-\tau^{\mu\nu}_{\alpha c,a}\cdot\tau^{\xi\zeta}_{\beta b,d}\right)g_{\xi\zeta}g^{\gamma\eta} \\ -\frac{\tau^{-1cd}{}_{\gamma\eta}}{2}\left(\tau^{\mu\nu}_{\alpha c,\beta}+\tau^{\mu\nu}_{\beta c,\alpha}-\tau^{\mu\nu}_{\alpha\beta,c}\right)\left(2\tau^{\xi\zeta}_{bd,a}-\tau^{\xi\zeta}_{ab,d}\right)g_{\xi\zeta}g^{\gamma\eta} \end{pmatrix}$$

The Ricci scalars then consequently read:

$$R^{*}=R^{*}{}_{\alpha\beta}G^{\alpha\beta}$$

$$=\begin{pmatrix} -\frac{\tau^{-1a}{}_{\chi}}{2}\left(\tau^{\mu}_{\alpha,ab}g_{\mu\beta}+\tau^{\mu}_{a,\alpha\beta}g_{\mu b}-\tau^{\mu}_{\alpha,a\beta}g_{\mu b}-\tau^{\mu}_{\beta,a\alpha}g_{\mu b}\right)g^{\chi b} \\ +\frac{\tau^{-1a}{}_{\chi}\tau^{-1c}{}_{\gamma}}{2}\left(\frac{1}{2}\tau^{\mu}_{a,\alpha}g_{\mu c}\cdot\tau^{\xi}_{b,\beta}g_{\xi d}+\tau^{\mu}_{\alpha,a}g_{\mu c}\cdot\tau^{\xi}_{\beta,b}g_{\xi d}-\tau^{\mu}_{\alpha,a}g_{\mu c}\cdot\tau^{\xi}_{\beta,d}g_{\xi b}\right)g^{\chi b}g^{\gamma d} \\ -\frac{\tau^{-1a}{}_{\chi}\tau^{-1c}{}_{\gamma}}{4}\left(\tau^{\mu}_{\alpha,\beta}g_{\mu c}+\tau^{\mu}_{\beta,\alpha}g_{\mu c}-\tau^{\mu}_{\alpha,c}g_{\mu\beta}\right)\left(2\tau^{\xi}_{b,a}g_{\xi d}-\tau^{\xi}_{a,d}g_{\xi b}\right)g^{\chi b}g^{\gamma d} \end{pmatrix}\tau^{-1\beta}{}_{\kappa}g^{\alpha\kappa}, \quad (148)$$

$$R^{*}=R^{*}{}_{\alpha\beta}G^{\alpha\beta}$$

$$=g_{\mu\nu}g^{\chi\delta}\frac{\tau^{-1ab}{}_{\chi\delta}}{2}\tau^{-1\alpha\beta}{}_{\lambda\kappa}g^{\lambda\kappa}\begin{pmatrix} -\left(\tau^{\mu\nu}_{\alpha\beta,ab}+\tau^{\mu\nu}_{ab,\alpha\beta}-\tau^{\mu\nu}_{\alpha b,a\beta}-\tau^{\mu\nu}_{\beta b,a\alpha}\right) \\ +\tau^{-1cd}{}_{\gamma\eta}\left(\frac{1}{2}\tau^{\mu\nu}_{ac,\alpha}\cdot\tau^{\xi\zeta}_{bd,\beta}+\tau^{\mu\nu}_{\alpha c,a}\cdot\tau^{\xi\zeta}_{\beta d,b}-\tau^{\mu\nu}_{\alpha c,a}\cdot\tau^{\xi\zeta}_{\beta b,d}\right)g_{\xi\zeta}g^{\gamma\eta} \\ -\frac{\tau^{-1cd}{}_{\gamma\eta}}{2}\left(\tau^{\mu\nu}_{\alpha c,\beta}+\tau^{\mu\nu}_{\beta c,\alpha}-\tau^{\mu\nu}_{\alpha\beta,c}\right)\left(2\tau^{\xi\zeta}_{bd,a}-\tau^{\xi\zeta}_{ab,d}\right)g_{\xi\zeta}g^{\gamma\eta} \end{pmatrix}. \quad (149)$$

4.4.1 A Simple Example: The Dirac Particle at Rest

We introduce a transformer matrix of the following type:

$$
\tau^{\mu\nu}_{\alpha\beta} = K^{\mu}_{\alpha} K^{\nu}_{\beta}
$$

$$
K^{\mu}_{\alpha} = \begin{pmatrix} F_0\left[f_0[t]\right] & 0 & 0 & 0 \\ 0 & F_1\left[f_1[t]\right] & 0 & 0 \\ 0 & 0 & F_2\left[f_2[t]\right] & 0 \\ 0 & 0 & 0 & F_3\left[f_3[t]\right] \end{pmatrix}. \tag{150}
$$

Then we assume a metric of constants with:

$$
g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & 0 & 0 \\ g_{01} & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix} \tag{151}
$$

and fix the wrapping functions F_i to:

$$
F_0\left[f_0[t]\right] = f_0[t]; \quad F_1\left[f_1[t]\right] = f_1[t]; \quad F_2\left[f_2[t]\right] = 1; \quad F_3\left[f_3[t]\right] = 1. \tag{152}
$$

Assuming a weak gravity condition:

$$
\delta G^{\alpha\beta} = G^{\alpha\beta}\cdot\delta_0 + \overbrace{G^{ab}\delta^{\alpha\beta}_{ab}}^{\text{Gravity}} \xrightarrow{\forall\, \delta^{\alpha\beta}_{ab} \ll \delta_0} = \frac{g^{\alpha\beta}}{F}\cdot\delta_0 , \tag{153}
$$

we can fulfill the field equations with just the setting $R^*=0$, which is to say:

$$
\begin{aligned}
0 = R^* &= R^*_{\alpha\beta} G^{\alpha\beta} \\
&= g_{\mu\nu} g^{\chi\delta} \frac{\tau^{-1ab}_{\chi\delta}}{2} \tau^{-1\alpha\beta}_{\lambda\kappa} g^{\lambda\kappa} \begin{pmatrix} -\left(\tau^{\mu\nu}_{\alpha\beta,ab} + \tau^{\mu\nu}_{ab,\alpha\beta} - \tau^{\mu\nu}_{\alpha b,a\beta} - \tau^{\mu\nu}_{\beta b,a\alpha}\right) \\ +\tau^{-1cd}_{\gamma\eta}\left(\frac{1}{2}\tau^{\mu\nu}_{ac,\alpha}\cdot\tau^{\xi\zeta}_{bd,\beta} + \tau^{\mu\nu}_{\alpha c,a}\cdot\tau^{\xi\zeta}_{\beta d,b} - \tau^{\mu\nu}_{\alpha c,a}\cdot\tau^{\xi\zeta}_{\beta b,d}\right) g_{\xi\zeta} g^{\gamma\eta} \\ -\frac{\tau^{-1cd}_{\gamma\eta}}{2}\left(\tau^{\mu\nu}_{\alpha c,\beta} + \tau^{\mu\nu}_{\beta c,\alpha} - \tau^{\mu\nu}_{\alpha\beta,c}\right)\left(2\tau^{\xi\zeta}_{bd,a} - \tau^{\xi\zeta}_{ab,d}\right) g_{\xi\zeta} g^{\gamma\eta} \end{pmatrix} . \\
&= \frac{2 g_{11}\left(f_0[t] f_1''[t] - f_0'[t] f_1'[t]\right)}{\left(\left(g_{01}\right)^2 - g_{00} g_{11}\right) f_0[t]^3 f_1[t]}
\end{aligned} \tag{154}
$$

Most interestingly, the solution can be obtained via a simple Dirac particle at rest approach with:

$$
f_0[t] = f_1[t] = C_{f1}\cdot e^{C_{f0}\cdot t}. \tag{155}
$$

Remembering that the Dirac theory [12] is based on a gravity free scalar equation, we might come to the conclusion that the metric equivalent of this theory could just be found via our approach here.

5 Conclusions

It was shown in the paper how the need for linearity of the quantum gravity field equations with respect to the volume scaling of the metric tensor provides us with a path to metrically obtain Dirac and Dirac-like field equations.

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7 Appendix

From Wikipedia, the free encyclopedia (https://en.wikipedia.org/wiki/Hamilton's_principle):

> *In physics, Hamilton's principle is William Rowan Hamilton's formulation of the principle of stationary action. It states that the dynamics of a physical system are determined by a variational problem for a functional based on a single function, the Lagrangian, which may contain all physical information concerning the system and the forces acting on it. The variational problem is equivalent to and allows for the derivation of the differential equations of motion of the physical system. Although formulated originally for classical mechanics, Hamilton's principle also applies to classical fields such as the electromagnetic and gravitational fields, and plays an important role in quantum mechanics, quantum field theory and criticality theories.*

So, the definition of the Hamilton principle is based on its “formulation of the principle of stationary action”. In simpler words, the variation of such an action should be zero or, mathematically formulated, should be put as follows:

$$\delta W = 0 = \delta \int_V d^n x \cdot \sqrt{-g} \cdot L \,. \tag{156}$$

Here L stands for the Lagrangian, W the action, and g gives the determinant of the metric tensor, which describes the system in question within an arbitrary Riemann space-time with the coordinates x. Thereby, we used the Hilbert formulation of the Hamilton principle [1] in a slightly more general form. We were able to show in [2] that the original Hilbert variation does not only produce the Einstein field equations [3] but also contains the Quantum Theory [2, 4, 5]. It should be noted that, while the original Hilbert paper [1] started with the Ricci scalar R as the integral kernel, which is to say L=R, we here used a general Langrangian, because—as we will show later in this appendix—this generality—in principle—is already contained inside the original Hilbert formulation. Even, as strange as it may sound at this point, general kernels with functions of the Ricci scalar f(R) [6] are already included (see [14]) in the Hilbert approach.

But what if we lived in a universe where the only thing that was certain was uncertainty?

One of the authors from [7], Dr. David Martin, always used the analogy of a moving fulcrum to demonstrate his uneasiness with the formulation (156) [13].

In [7] we were able to show that the Hamilton principle itself hinders us to localize any system or object at a certain position. We also see that this contradicts the concept of particles. Everything seems to be permanently on the move or—rather—ever-jittering.

But if this ever-jittering fulcrum was one of the fundamental properties of our universe, should we then not take this into account when formulating the laws of this very universe? Shouldn't we better write (156) as follows:

$$\delta W \rightarrow 0 \cong \delta \int_V d^n x \cdot \sqrt{-g} \cdot L \, ? \tag{157}$$

And while we are at it, should we not start to investigate an even more general principle like:

$$\delta W \rightarrow f\left(W, x, g_{\alpha\beta}\right) = \delta \int_V d^n x \cdot \sqrt{-g} \cdot L \, ? \tag{158}$$

The interesting aspect about this is that this investigation was already—partially—done by (surprise, surprise) e.g., Hilbert and Einstein. But instead of introducing and explaining it in this way, they have "hidden" their generalization inside other concepts like the introduction of a cosmological constant or— oh yes—the postulation of matter and its introduction via an ominous and purely postulated parameter L_M, which is to say, a Lagrange matter term. Thereby, as we should explicitly point out here, the "hiding" never was intentionally, but just caused by the knowledge and understanding at the time.

7.1 The Classical Hamilton Extremal Principle and How to Obtain Einstein's General Theory of Relativity with Matter (!) and Quantum Theory… Also with Matter (!)

The famous German mathematician David Hilbert [1], even though applying his technique only to derive the Einstein field equations for the General Theory of Relativity [3] in four dimensions,—in principle—extended the classical Hamilton principle to an arbitrary Riemann space-time with a very general variation by not only—as Hamilton and others had done—concentrating on the evolution of the given problem or system in time, but with respect to all its dimensions. His formulation of the Hamilton extremal principle looked as follows:

$$\delta W = 0 = \delta \int_V d^n x \left(\sqrt{-g} \cdot \left(R - 2\Lambda + L_M \right) \right). \tag{159}$$

There we have the Ricci scalar of curvature R, the cosmological constant Λ, the Lagrange density of matter L_M, and the determinant g of the metric tensor of the Riemann space-time $g_{\alpha\beta}$. For historical reasons, it should be mentioned that Hilbert's original work [1] did not contain the cosmological constant because it was added later by Einstein in order to obtain a static universe, but this is not of any importance here. The evaluation of the so-called Einstein-Hilbert action (159) brought indeed the Einstein General Theory of Relativity [3], but it did not produce the other great theory physicists have found, which is the Quantum Theory. It was not before the author of this article here, about one hundred years after the publication of Hilbert's paper [1], extended Hilbert's approach by considering scaling factors to the metric tensor and showed that Quantum Theory already resides inside the sufficiently general General Theory of Relativity [2, 4, 7, 8, 9, 10]. We will not discuss the reason why this simple idea has not been tried out by other scientists before, but we may still express our amazement about the fact that a simple extension of the type:

$$G_{\alpha\beta} = g_{\alpha\beta} \cdot F[f] \tag{160}$$

solves one of the greatest problems in science[2], namely the unification of physics and that it took science more than 100 years to come up with the idea. Using the symbol G for the determinant of the scaled metric tensor $G_{\alpha\beta}$ from (1) of the Riemann space-time, we can rewrite the Einstein-Hilbert action from (159) as follows:

$$\delta W = 0 = \delta \int_V d^n x \left(\sqrt{-G} \cdot F^q \cdot \left(R^* - 2\Lambda + L_M \right) \right) \tag{161}$$

where we even used another generalization, namely the kernel extension F^q, which could also be possible and still converges to the classical form for F→1. Here, which is to say in this paper, we will only consider examples with q=0, but for completeness and later investigation we shall mention that a comprehensive consideration of variational integrals for the cases of general q are to be found in [4]. Performing the variation in (161) with respect to the metric $G_{\alpha\beta}$ and remembering that the Ricci curvature of such a metric (e.g., [7], appendix D) changes the whole variation to:

$$\begin{aligned} \delta W = 0 &= \delta \int_V d^n x \left(\sqrt{-G} \cdot F^q \cdot \left(R^* - 2\Lambda + L_M \right) \right) \\ &= \delta \int_V d^n x \left(\sqrt{-G} \cdot F^q \cdot \left(\left(\begin{matrix} \frac{R}{F} - \frac{1}{2F^2} \left(\begin{matrix} (n-1)\left(\overbrace{2g^{ab}F_{,ab} + F_{,d}g^{cd}g^{ab}g_{ab,c}}^{=2\Delta F - 2F_{,d}g^{cd}{}_{,c}} \right) \\ -nF_{,d}g^{cd}g^{ab}g_{ac,b} \end{matrix} \right) \\ -(n-1)\frac{g^{ab}F_{,a} \cdot F_{,b}}{4F^3}(n-6) \end{matrix} \right) - 2\Lambda + L_M \right) \right), \end{aligned} \tag{162}$$

results in:

[2] This does not mean, of course, that we should not also look out for generalizations of the scaled metric and investigate those as we did in [10].

$$
\begin{aligned}
0 &= \left(R^{*}{}_{\alpha\beta} - \frac{1}{2} R^{*} \cdot G_{\alpha\beta} \right) \overbrace{\left(\frac{1}{F} \cdot \delta g^{\alpha\beta} + g^{\alpha\beta} \cdot \delta\left(\frac{1}{F} \right) \right)}^{\delta G^{\alpha\beta}} \\
&= \left[\left(R_{\alpha\beta} - \frac{1}{2F} \left(\begin{array}{c} F_{,\alpha\beta}(n-2) + F_{,ab} g_{\alpha\beta} g^{ab} + F_{,a} g^{ab} \left(g_{\beta b,\alpha} - g_{\beta\alpha,b} \right) - \\ F_{,\alpha} g^{ab} g_{\beta b,a} - F_{,\beta} g^{ab} g_{\alpha b,a} + F_{,d} g^{cd} \left(\begin{array}{c} g_{\alpha c,\beta} - \frac{1}{2} n g_{\alpha c,\beta} - \frac{1}{2} n g_{\beta c,\alpha} \\ + \frac{1}{2} n g_{\alpha\beta,c} + \frac{1}{2} g_{\alpha\beta} g_{ab,c} g^{ab} \end{array} \right) \end{array} \right) \right. \right. \\
&\qquad \left. + \frac{1}{4F^2} \left(F_{,\alpha} \cdot F_{,\beta} (3n-6) + g_{\alpha\beta} F_{,c} F_{,d} g^{cd} (4-n) \right) \right) \\
&\qquad \left. + (n-1) \left(\frac{1}{2F} \left(\begin{array}{c} \left(\overbrace{2 g^{ab} F_{,ab} + F_{,d} g^{cd} g^{ab} g_{ab,c}}^{=2\Delta F - 2 F_{,d} g^{cd}{}_{,c}} \right) \\ - \frac{n}{(n-1)} F_{,d} g^{cd} g^{ab} g_{ac,b} \end{array} \right) + \frac{g^{ab} F_{,a} \cdot F_{,b}}{4F^2} (n-6) - \frac{R}{(n-1)} \right) \cdot \frac{g_{\alpha\beta}}{2} \right] \delta G^{\alpha\beta}
\end{aligned}
\quad ,(163)
$$

when setting q=0 and assuming a vanishing cosmological constant. With a cosmological constant we have to write:

$$
\begin{aligned}
0 &= \left(R^{*}{}_{\alpha\beta} - \frac{1}{2} R^{*} \cdot G_{\alpha\beta} \right) \overbrace{\left(\frac{1}{F} \cdot \delta g^{\alpha\beta} + g^{\alpha\beta} \cdot \delta\left(\frac{1}{F} \right) \right)}^{\delta G^{\alpha\beta}} \\
&= \left(\begin{array}{c} \boxed{\boxed{R_{\alpha\beta} - R \frac{g_{\alpha\beta}}{2}}} + \boxed{\Lambda \cdot g_{\alpha\beta}} \\
- \frac{1}{2F} \left(\begin{array}{c} F_{,\alpha\beta}(n-2) + F_{,ab} g_{\alpha\beta} g^{ab} + F_{,a} g^{ab} \left(g_{\beta b,\alpha} - g_{\beta\alpha,b} \right) - \\ F_{,\alpha} g^{ab} g_{\beta b,a} - F_{,\beta} g^{ab} g_{\alpha b,a} + F_{,d} g^{cd} \left(\begin{array}{c} g_{\alpha c,\beta} - \frac{1}{2} n g_{\alpha c,\beta} - \frac{1}{2} n g_{\beta c,\alpha} \\ + \frac{1}{2} n g_{\alpha\beta,c} + \frac{1}{2} g_{\alpha\beta} g_{ab,c} g^{ab} \end{array} \right) \end{array} \right) \delta G^{\alpha\beta} \\
+ \frac{1}{4F^2} \left(F_{,\alpha} \cdot F_{,\beta} (3n-6) + g_{\alpha\beta} F_{,c} F_{,d} g^{cd} (4-n) \right) \\
+ (n-1) \left(\frac{1}{2F} \left(\begin{array}{c} 2\Delta F - 2 F_{,d} g^{cd}{}_{,c} \\ - \frac{n}{(n-1)} F_{,d} g^{cd} g^{ab} g_{ac,b} \end{array} \right) + \frac{g^{ab} F_{,a} \cdot F_{,b}}{4F^2} (n-6) \right) \cdot \frac{g_{\alpha\beta}}{2}
\end{array} \right)
\end{aligned}
\quad . \qquad (164)
$$

For better recognition of the classical terms, we have reordered a bit and boxed the classical vacuum part of the Einstein field equations (double lines) and the cosmological constant term (single line). Everything else can be—no, represents (!)—matter or quantum effects or both.

Thus, we also—quite boldly—have set the matter density L_M equal to zero, because we see that already our simple metric scaling brings in quite some options for the construction of matter. It will be shown elsewhere [10] that there is much more which is based on the same technique.

7.2 The Principle of the Ever-Jittering Fulcrum and the Alternate Hamilton Principle

We might bring forward two reasons why we could doubt the fundamentality of the Hamilton principle even in its most general form of the generalized Einstein-Hilbert action:

a) The principle was postulated and never fundamentally derived.
b) Even the formulation of this principle in its classical form (159) results in a variety of options where factors, constants, kernel adaptations, etc. could be added, so that the rigid setting of the integral to zero offers some doubt in itself. A calculation process which offers a variety of add-ons and options should not contain such a dogma. The result should be kept open and general. Dr. David Martin proposed this as the "tragedy of the jittering fulcrum" and we therefore named this principle "David's principle of the ever-jittering fulcrum" [13]. It demands:

$$\begin{aligned}\delta_{g_{\alpha\beta}} W \simeq ? \simeq \delta_{g_{\alpha\beta}} \int_V d^n x \sqrt{-g} \times R \\ \delta_{G_{\alpha\beta}} W \simeq ? \simeq \delta_{G_{\alpha\beta}} \int_V d^n x \sqrt{-G} \times R^* \end{aligned} . \tag{165}$$

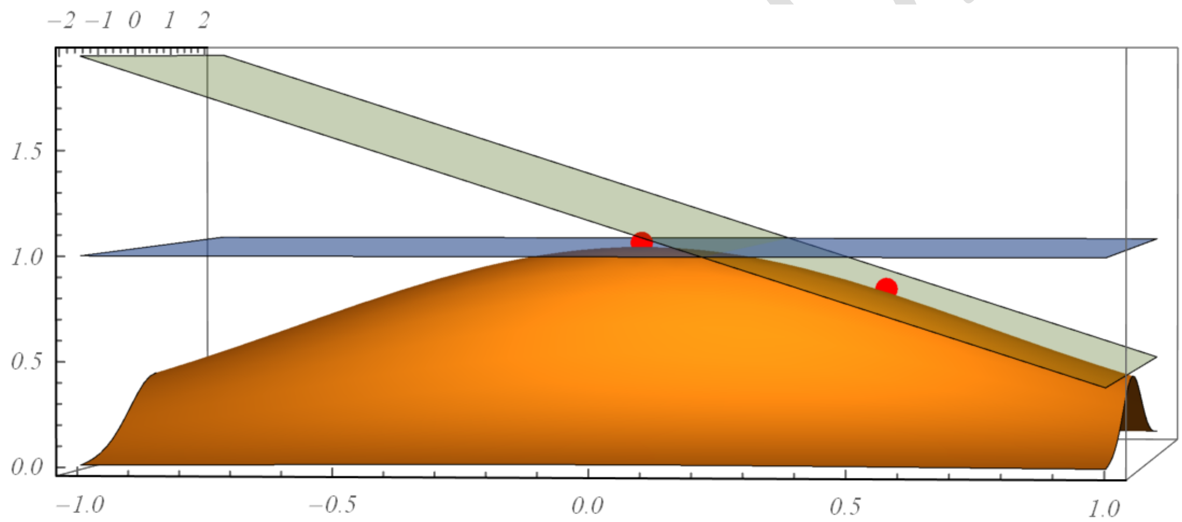

Fig. A1: David's principle of the ever-jittering fulcrum cannot accept a dogmatic insistence on a zero outcome of the Einstein-Hilbert action (159) or (generalized and also bringing about the Quantum Theory) (161). Instead it should allow for all states and not just the extremal position (see the two red dots and the corresponding tangent planes in the picture).

One of the simplest generalizations of the classical principle could be the linear one, which is illustrated in figure A1. It could be constructed as follows:

$$\int_V d^n x \sqrt{-g} \times \not{K}^{\alpha\beta} \cdot g_{\alpha\beta} = \delta_{g_{\alpha\beta}} W = \delta_{g_{\alpha\beta}} \int_V d^n x \sqrt{-g} \times R \, . \tag{166}$$

Thereby we have used the classical form with the unscaled metric tensor, respectively without setting the factor apart from the rest of the metric. Performing of the variation on the right-hand side and setting

$$\not{K}^{\alpha\beta} = H \cdot \delta g^{\alpha\beta} \tag{167}$$

or—for the reason of—maximum generality even:

$$\not{H}^{\alpha\beta} = H_{ab}^{\alpha\beta} \cdot \delta\gamma^{ab} = H \cdot \delta g^{\alpha\beta} \tag{168}$$

just gives us the same result as we would obtain it when assuming a non-zero cosmological constant, because evaluation yields:

$$\begin{gathered}\int_V d^n x\sqrt{-g} \times H \cdot \delta g^{\alpha\beta} \cdot g_{\alpha\beta} = \int_V d^n x\sqrt{-g} \times \left(R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} \right)\delta g^{\alpha\beta} \\ \Rightarrow 0 = \int_V d^n x\sqrt{-g} \times \left(R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} - H g_{\alpha\beta} \right)\delta g^{\alpha\beta}\end{gathered}, \tag{169}$$

respectively:

$$\begin{gathered}\int_V d^n x\sqrt{-g} \times H_{ab}^{\alpha\beta} \cdot \delta\gamma^{ab} \cdot g_{\alpha\beta} = \int_V d^n x\sqrt{-g} \times \left(R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} \right)\delta g^{\alpha\beta} \\ \Rightarrow 0 = \int_V d^n x\sqrt{-g} \times \left(R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} - H g_{\alpha\beta} \right)\delta g^{\alpha\beta}\end{gathered}. \tag{170}$$

Simply setting *H*=-Λ (c.f. single-line boxed term in equation (164)) demonstrates this.

Nothing else is the usage of a general functional term T, being considered a function of the coordinates of the system (perhaps even the metric tensor) in a general manner, as follows:

$$\int_V d^n x\sqrt{-g} \times T = \delta_{g_{\alpha\beta}} W = \delta_{g_{\alpha\beta}} \int_V d^n x\sqrt{-g} \times R . \tag{171}$$

As before, performing of the variation on the right-hand side and setting

$$T = T_{\alpha\beta} \cdot \delta g^{\alpha\beta} \tag{172}$$

gives us something which was classically postulated under the variational integral, namely the classical energy-matter tensor. This time, however, it simply pops up as a result of David's principle of the jittering fulcrum and is equivalent to the introduction of the term L_M under the variational integral. Evaluation yields:

$$\begin{gathered}\int_V d^n x\sqrt{-g} \cdot T_{\alpha\beta} \cdot \delta g^{\alpha\beta} = \int_V d^n x\sqrt{-g} \times \left(R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} \right)\delta g^{\alpha\beta} \\ \Rightarrow 0 = \int_V d^n x\sqrt{-g} \times \left(R_{\alpha\beta} - \frac{R}{2} g_{\alpha\beta} - T_{\alpha\beta} \right)\delta g^{\alpha\beta}\end{gathered}. \tag{173}$$

So, we see that in introducing a cosmological constant and in postulating a matter term, even Einstein and Hilbert already—in principle—"experimented" with a non-extremal setting for the Hamilton extremal principle.

Apart from linear dependencies and other functions or functional terms, we could just assume a general outcome like:

$$f(W) = f\left(\int_V d^n x\sqrt{-g} \times R \right) = \delta_{g_{\alpha\beta}} W = \delta_{g_{\alpha\beta}} \int_V d^n x\sqrt{-g} \times R \cdot \tag{174}$$

This, however, would not give us any substantial hint where to move on, respectively, which of the many possible paths to follow. We therefore here start our investigation with the assumption of an eigen result for the variation as follows:

$$\not{H} \cdot W = \not{H} \cdot \int_V d^n x\sqrt{-g} \times R = \delta_{g_{\alpha\beta}} W = \delta_{g_{\alpha\beta}} \int_V d^n x\sqrt{-g} \times R . \tag{175}$$

This leads to:

$$\int_V d^n x\sqrt{-g}\left(R_{\kappa\lambda}\delta g^{\kappa\lambda}-R\cdot\left(\frac{1}{2}\cdot g_{\kappa\lambda}\delta g^{\kappa\lambda}+\not{h}\right)\right)=0. \qquad (176)$$

As the term $\not{h}$ could always be expanded into an expression like:

$$\not{h}=H\cdot g_{\kappa\lambda}\delta g^{\kappa\lambda}, \qquad (177)$$

we obtain from (176):

$$\begin{aligned}0&=\int_V d^n x\sqrt{-g}\left(R_{\kappa\lambda}\delta g^{\kappa\lambda}-R\cdot\left(\frac{1}{2}+H\right)g_{\kappa\lambda}\delta g^{\kappa\lambda}\right)\\&=\int_V d^n x\sqrt{-g}\left(R_{\kappa\lambda}-R\cdot\left(\frac{1}{2}+H\right)g_{\kappa\lambda}\right)\delta g^{\kappa\lambda}\\&\Rightarrow R_{\kappa\lambda}-R\cdot\left(\frac{1}{2}+H\right)g_{\kappa\lambda}=0\end{aligned} \qquad (178)$$

We realize that the term *H* can be a general scalar even if we would demand the term $\not{h}$ to be a constant.

The complete equation when assuming a scaled metric tensor of the form (1) would read:

$$\left(\begin{array}{c}\left(\begin{array}{c}R_{\alpha\beta}-\frac{1}{2F}\left(\begin{array}{l}F_{,\alpha\beta}(n-2)+F_{,ab}g_{\alpha\beta}g^{ab}\\+F_{,a}g^{ab}\left(g_{\beta b,\alpha}-g_{\beta\alpha,b}\right)-F_{,\alpha}g^{ab}g_{\beta b,a}-F_{,\beta}g^{ab}g_{\alpha b,a}\\+F_{,d}g^{cd}\left(g_{\alpha c,\beta}-\frac{1}{2}ng_{\alpha c,\beta}-\frac{1}{2}ng_{\beta c,\alpha}+\frac{1}{2}ng_{\alpha\beta,c}+\frac{1}{2}g_{\alpha\beta}g_{ab,c}g^{ab}\right)\end{array}\right)\\+\frac{1}{4F^2}\left(F_{,\alpha}\cdot F_{,\beta}(3n-6)+g_{\alpha\beta}F_{,c}F_{,d}g^{cd}(4-n)\right)\end{array}\right)\\-\left(\begin{array}{c}R-\frac{1}{2F}\left((n-1)\left(\overbrace{2g^{ab}F_{,ab}+F_{,d}g^{cd}g^{ab}g_{ab,c}}^{=2\Delta F-2F_{,d}g^{cd}{}_{,c}}\right)-nF_{,d}g^{cd}g^{ab}g_{ac,b}\right)\\-(n-1)\frac{g^{ab}F_{,a}\cdot F_{,b}}{4F^2}(n-6)\end{array}\right)\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\end{array}\right)=0, \qquad (179)$$

and in the case of metrics with constant components this equation simplifies to:

$$\left(\begin{array}{c}\left(\begin{array}{c}R_{\alpha\beta}-\frac{1}{2F}\left(F_{,\alpha\beta}(n-2)+F_{,ab}g_{\alpha\beta}g^{ab}\right)\\+\frac{1}{4F^2}\left(F_{,\alpha}\cdot F_{,\beta}(3n-6)+g_{\alpha\beta}F_{,c}F_{,d}g^{cd}(4-n)\right)\end{array}\right)\\-\left(R-\frac{(n-1)}{2F}\left(2g^{ab}F_{,ab}+\frac{g^{ab}F_{,a}\cdot F_{,b}}{2F}(n-6)\right)\right)\cdot\left(\frac{1}{2}+H\right)g_{\alpha\beta}\end{array}\right)=0. \qquad (180)$$

7.2.1 The Question of Stability

From purely mechanical considerations, one might assume that extremal solutions of the variational equation (165) correspond to more stable states than non-extremal solutions, and in fact we will find this in connection with the 3-generations problem, which we have discussed in [12].

7.3 References of the Appendix

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